International Journal of Physical Sciences

Volume 9 Number 19 16 October, 2014

ISSN 1992-1950

Academic Journals

ABOUT IJPS

The **International Journal of Physical Sciences (IJPS)** is published weekly (one volume per year) by Academic Journals.

International Journal of Physical Sciences (IJPS) is an open access journal that publishes high-quality solicited and unsolicited articles, in English, in all Physics and chemistry including artificial intelligence, neural processing, nuclear and particle physics, geophysics, physics in medicine and biology, plasma physics, semiconductor science and technology, wireless and optical communications, materials science, energy and fuels, environmental science and technology, combinatorial chemistry, natural products, molecular therapeutics, geochemistry, cement and concrete research, metallurgy, crystallography and computer-aided materials design. All articles published in IJPS are peer-reviewed.

Contact Us

Editorial Office: ijps@academicjournals.org

Help Desk: helpdesk@academicjournals.org

Website: http://www.academicjournals.org/journal/IJPS

Submit manuscript online http://ms.academicjournals.me/

Editors

Editorial Board

International Journal of Physical Sciences

Table of Contents: Volume 9 Number 19, 16 October, 2014

ARTICLES

academicJournals

Vol. 9(19), pp. 415-429, 16 October, 2014
DOI: 10.5897/IJPS2014.4178
ISSN 1992 - 1950
Article Number: C712A7748331

http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

On solving the nonlinear Schrödinger-Boussinesq equation and the hyperbolic Schrödinger equation by using the $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method

E. M. E. Zayed* and K. A. E. Alurrfi

Department of Mathematics, Faculty of Science, Zagazig University, P. O. Box 44519. Zagazig, Egypt.

Received 23 June, 2014; Accepted 15 September, 2014

The propagation of the optical solitons is usually governed by the nonlinear Schrödinger equations. In this article, the two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method is employed to construct the exact traveling wave solutions with parameters of two nonlinear partial differential equations (PDEs) namely, the (1+1)-dimensional nonlinear Schrödinger-Boussinesq system and the (2+1)-dimensional hyperbolic nonlinear Schrödinger (HNLS) equation which describe the propagation of optical pulses in optic fibers. When the parameters are replaced by special values, the solitary wave solutions of these equations are found from the traveling waves.

Key words: The two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method, nonlinear Schrödinger-Boussinesq system, hyperbolic nonlinear Schrödinger (HNLS) equation, exact traveling wave solutions, solitary wave solutions.

INTRODUCTION

In the recent years, investigations of exact solutions to nonlinear partial differential equation (PDEs) play an important role in the study of nonlinear physical phenomena. Many powerful methods have been presented, such as the inverse scattering method (Ablowitz and Clarkson, 1991), the Hirota bilinear transform method (Hirota, 1971), the truncated Painleve expansion method (Weiss et al., 1983; Kudryashov, 1988, 1990, 1991), the Backlund transform method (Miura, 1978; Rogers and Shadwick, 1982), the exp-function method (He and Wu, 2006; Yusufoglu, 2008; Zhang, 2008; Bekir, 2009, 2010), the tanh-function method (Abdou 2010; Fan, 2000; Zhang and Xia, 2008; Yusufoglu and Bekir, 2008), the Jacobi elliptic function expansion method (Chen and Wang, 2005; Liu et al., 2001; Lu, 2005), the $\left(\frac{G'}{G}\right)$-expansion method (Wang et al., 2008; Zhang et al., 2008; Zayed and Gepreel, 2009; Zayed, 2009; Bekir, 2008; Ayhan and Bekir, 2012; Kudryashov, 2010a, b; Aslan, 2010; Zayed, 2010;), the modified $\left(\frac{G'}{G}\right)$-expansion method (Zhang et al., 2011),

*Corresponding author. E-mail: e.m.e.zayed@hotmail.com; alurrfi@yahoo.com

PACS: 02.30.Jr; 05.45.Yv; 02.30.Ik.

the $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method (Li et al., 2010; Zayed and Abdelaziz, 2012; Zayed et al., 2012; Zayed and Alurrfi, 2014a, b), the Riccati equation method (Ma and Fuchssteiner, 1996), the bilinear method (Ma, 2011, 2013), the transformed rational function method (Ma and Lee, 2009), the multiple exp-function method (Ma and Zhu, 2012) and so on.

The key idea of the one variable $\left(\frac{G'}{G}\right)$-expansion method is that the exact solutions of nonlinear PDEs can be expressed by a polynomial in one variable $\left(\frac{G'}{G}\right)$ in which $G=G(\xi)$ satisfies the second order linear ordinary differential equation (ODE) $G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0$, where λ, μ are constants and $'=\frac{d}{d\xi}$. The key idea of the two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method is that the exact traveling wave solutions of nonlinear PDEs can be expressed by a polynomial in two variables $\left(\frac{G'}{G}\right)$ and $\left(\frac{1}{G}\right)$ in which $G=G(\xi)$ satisfies the second order linear ODE $G''(\xi)+\lambda G(\xi)=\mu$, where λ and μ are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear PDEs. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using this method. Recently, Li et al. (2010) have applied the $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method and determined the exact solutions of the nonlinear Zakharov equations, while Zayed and Abdelaziz (2012), Zayed et al. (2012), and Zayed and Alurrfi (2014a, b), respectively have used this method to find the exact solutions of the nonlinear combined KdV-mKdV equation, the nonlinear Kadomtsev-Petviashvili equation, the nonlinear PDE for nanobioscinces and two higher order nonlinear evolution equations namely, the nonlinear Klein-Gordon equations and the nonlinear Pochhammer-Chree equations .

The objective of this paper is to apply the two variables $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method obtained in Li et al. (2010), Zayed and Abdelaziz (2012), Zayed et al. (2012), and Zayed and Alurrfi (2014a,b) to find the exact traveling wave solutions of the following two different nonlinear equations which are not yet discussed:

(i) The (1+1)-dimensional Schrödinger-Boussinesq system (SB-system) (Kilicman and Abazari, 2012):

$$\begin{aligned} &iu_t+u_{xx}-auv=0,\\ &v_{tt}-v_{xx}+v_{xxxx}-b(|u|^2)_{xx}=0, \end{aligned} \quad (1)$$

Where $t>0$, $x\in[0,L]$, for some $L>0$, and a,b are real constants. Here, *u* and *v* are, respectively a complex-valued and a real-valued function.

(ii) The (2+1)-dimensional hyperbolic nonlinear Schrödinger (HNLS) equation (Fen, 2012):

$$iu_y+\frac{1}{2}u_{xx}-\frac{1}{2}u_{tt}+|u|^2u=0, \quad (2)$$

Where *u(x,y,t)* is a complex-valued function which represents the slowly varying envelope of propagation, x is the dimensionless variable, y is the propagation coordinate and t is the time.

The SB-system (Equation 1) is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma (Makhankov, 1974) and diatomic lattice system (Yajima and Satsuma, 1979). The SB-system has been discussed in Kilicman and Abazari (2012) using the $\left(\frac{G'}{G}\right)$-expansion method and its exact solutions has been found. Equation 2 can be derived from optics (Gorz and Haelterman, 2008) and large-scale Rossby waves (Tan and Wu, 1993). Various types of HNLS equations describing time and space evolutions of slowly varying envelopes have wide applications in various branches of physics (Tang and Shukla 2007; Li, 2007). HNLS equation has been investigated in Fen (2012) using the theory of bifurcations of dynamical system and its exact solutions have been presented.

DESCRIPTION OF THE TWO VARIABLE $\left(\frac{G'}{G},\frac{1}{G}\right)$-EXPANSION METHOD

Before the main steps of this method are described, the following remarks are needed (Li et al., 2010; Zayed and Abdelaziz, 2012; Zayed et al., 2012; Zayed and Alurrfi, 2014a, b):

Remark 1

If the second order linear ODE is considered:

$$G''(\xi)+\lambda G(\xi)=\mu, \quad (3)$$

and set $\phi=\frac{G'}{G}$, $\psi=\frac{1}{G}$, then we get:

$$\phi' = -\phi^2 + \mu\psi - \lambda,\ \psi' = -\phi\psi. \tag{4}$$

Where λ and μ are constants while $' = \frac{d}{d\xi}$.

Remark 2

If $\lambda < 0$, then the general solution of Equation 3 has the form:

$$G(\xi) = A_1 \sinh(\xi\sqrt{-\lambda}) + A_2 \cosh(\xi\sqrt{-\lambda}) + \tfrac{\mu}{\lambda}, \tag{5}$$

Where A_1 and A_2 are arbitrary constants. Consequently, we have

$$\psi^2 = -\frac{\lambda}{\lambda^2\sigma_1 + \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{6}$$

Where $\sigma_1 = A_1^2 - A_2^2$

Remark 3

If $\lambda > 0$, then the general solution of Equation 3 has the form:

$$G(\xi) = A_1 \sin(\xi\sqrt{\lambda}) + A_2 \cos(\xi\sqrt{\lambda}) + \tfrac{\mu}{\lambda}, \tag{7}$$

and hence

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma_2 - \mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{8}$$

Where $\sigma_2 = A_1^2 + A_2^2$

Remark 4

If $\lambda = 0$, then the general solution of Equation 3 has the form:

$$G(\xi) = \tfrac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{9}$$

and hence

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}(\phi^2 - 2\mu\psi). \tag{10}$$

Suppose we have the following nonlinear evolution equation.

$$F(u, u_t, u_x, u_{xx}, ...) = 0, \tag{11}$$

Where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, the main steps of the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method are given (Li et al., 2010; Zayed and Abdelaziz, 2012; Zayed et al., 2012; Zayed and Alurrfi, 2014a, b):

Step 1

The traveling wave transformation

$$u(x,t) = u(\xi), \qquad \xi = x - Ct, \tag{12}$$

Where C is a constant, reduces Equation 11 to an ODE in the form:

$$P(u, u', u'', ...) = 0, \tag{13}$$

Where *P* is a polynomial of $u(\xi)$ and its total derivatives with respect to ξ.

Step 2

Assuming that the solution of Equation 13 can be expressed by a polynomial in the two variables ϕ and ψ as follows:

$$u(\xi) = a_0 + \sum_{i=1}^{N} a_i \phi^i + \sum_{i=1}^{N} b_i \phi^{i-1}\psi, \tag{14}$$

Where a_0, a_i and b_i $(i = 1, 2, ..., N)$ are constants to be determined later satisfying $a_N^2 + b_N^2 \neq 0$.

Step 3

Determine the positive integer N in Equation 14 by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in Equation 13. More precisely we define the degree of $u(\xi)$ as $D[u(\xi)] = N$ which gives rise to the degree of other expressions as follows:

$$D\left[\frac{d^q u}{d\xi^q}\right] = N + q,$$
$$D\left[u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right] = Np + s(q+N). \qquad (15)$$

Therefore, we can get the value of N in Equation 14.

Step 4

Substitute Equation 14 into Equation 13 along with Equations 4 and 6, the left- hand side of Equation 13 can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is not longer than 1. Equating each coefficients of this polynomial to 0, yields a system of algebraic equations which can be solved by using the Maple or Mathematica to get the values of $a_i, b_i, C, \mu, A_1, A_2$ and λ where $\lambda < 0$. Similarly, substitute Equation 14 into Equation 13 along with Equations 4 and 8 for $\lambda > 0$ or Equations 4 and 10 for $\lambda = 0$, we obtain the exact solutions of Equation 13 expressed by hyperbolic functions, trigonometric functions and rational functions, respectively.

APPLICATIONS

Here, the method described earlier is applied to find the exact traveling wave solutions of Equations 1 and 2 which are very important in the mathematical physics and have been paid attention by many researchers.

Example 1

The (1+1)-dimensional nonlinear SB-system (Equation 1)

We start with the (1+1)-dimensional nonlinear SB-system (Equation 1). Assume that the solution of Equation 1 can be written as:

$$u(x,t) = U(\xi)e^{i\eta},$$
$$v(x,t) = V(\xi), \qquad (16)$$

Where $\xi = kx + \omega t$, $\eta = px + qt$ and k, ω, p, q are constants, $i = \sqrt{-1}$. Substituting Equation 16 into Equation 1, we have the following system of nonlinear ODEs:

$$k^2 U'' + i(2kp+\omega)U' - aUV - (p^2+q)U = 0, \qquad (17)$$

$$(\omega^2 - k^2)V'' + k^4 V^{(4)} - bk^2(U^2)'' = 0. \qquad (18)$$

Integrating Equation 18 twice and taking integration constants to be 0, the Equations 17 to 18 reduces to the following system:

$$k^2 U'' + i(2kp+\omega)U' - aUV - (p^2+q)U = 0, \qquad (19)$$

$$(\omega^2 - k^2)V + k^4 V'' - bk^2 U^2 = 0. \qquad (20)$$

Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Equations 19 and 20, we obtain $N = M = 2$. Consequently, Equations 19 and 20 have the formal solutions:

$$U(\xi) = \alpha_0 + \alpha_1\phi(\xi) + \alpha_2\phi^2(\xi) + \beta_1\psi(\xi) + \beta_2\phi(\xi)\psi(\xi) \qquad (21)$$

$$V(\xi) = c_0 + c_1\phi(\xi) + c_2\phi^2(\xi) + d_1\psi(\xi) + d_2\phi(\xi)\psi(\xi) \qquad (22)$$

Where $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, c_0, c_1, c_2, d_1$ and d_2 are constants to be determined later satisfying $\alpha_2^2 + \beta_2^2 \neq 0$, $c_2^2 + d_2^2 \neq 0$. There are three cases to be discussed as follows:

Case 1: Hyperbolic function solutions ($\lambda < 0$)

If $\lambda < 0$, substituting Equations 21 and 22 into Equations 19 and 20 and using Equations 4 and 6, the left-hand sides are converted into polynomial in ϕ and ψ. Setting each coefficient of this polynomial to 0, yields a system of algebraic equations in $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, c_0, c_1, c_2, d_1, d_2, \mu, \lambda, \omega, k, p$ and q as follows:

$$-a\alpha_2 c_2 + 6k^2\alpha_2 + \frac{a\beta_2 d_2 \lambda}{\lambda^2\sigma_1 + \mu^2} = 0,$$

$$-2i(2kp+\omega)\alpha_2 + 2k^2\alpha_1 - a\alpha_1 c_2 - a\alpha_2 c_1 - \frac{\lambda}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1 d_2 - a\beta_2 d_1 - 6k^2\beta_2\mu\right) = 0,$$

$$-a\alpha_2 d_2 + 6k^2\beta_2 - a\beta_2 c_2 = 0,$$

$$-(p^2+q)\alpha_2 - i(2kp+\omega)\alpha_1 - a\alpha_0 c_2 - a\alpha_1 c_1 - a\alpha_2 c_0 + 8k^2\alpha_2\lambda + \frac{\lambda^2 a\beta_2 d_2}{\lambda^2\sigma_1+\mu^2}$$
$$-\frac{\lambda}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1 d_1 + i(2kp+\omega)\beta_2\mu + k^2\left(2\alpha_2\mu^2 - \beta_1\mu\right)\right) = 0,$$
$$-2i(2kp+\omega)\beta_2 - a\alpha_1 d_2 - a\alpha_2 d_1 - k^2\left(10\alpha_2\mu - 2\beta_1\right) - a\beta_1 c_2 - a\beta_2 c_1 - \frac{2\lambda\mu a\beta_2 d_2}{\lambda^2\sigma_1+\mu^2} = 0,$$

$$-(p^2+q)\alpha_1-2i\,(2kp+\omega)\alpha_2\lambda-a\alpha_0c_1-a\alpha_1c_0+2k^2\alpha_1\lambda+\frac{\lambda^2}{\lambda^2\sigma_1+\mu^2}\left(a\beta_1d_2+a\beta_2d_1+6k^2\beta_2\mu\right)=0,$$

$$-a\alpha_0d_2-a\alpha_1d_1+k^2\left(-3\alpha_1\mu+5\beta_2\lambda\right)-a\beta_1c_1-a\beta_2c_0-(p^2+q)\beta_2+i\,(2kp+\omega)\left(-\beta_1+2\alpha_2\mu\right)+\frac{2\lambda\mu}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1d_2-a\beta_2d_1-6k^2\beta_2\mu\right)=0,$$

$$k^2\lambda\left(\beta_1-4\alpha_2\mu\right)+i\,(2kp+\omega)\left(-\beta_2\lambda+\alpha_1\mu\right)-a\alpha_0d_1-(p^2+q)\beta_1-a\beta_1c_0+\frac{2\lambda\mu}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1d_1+i\,(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)=0,$$

$$-i\,(2kp+\omega)\lambda\alpha_1+2k^2\alpha_2\lambda^2-a\alpha_0c_0-\frac{\lambda^2}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1d_1+i\,(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)-(p^2+q)\alpha_0=0,$$

$$6k^4c_2-bk^2\alpha_2^2+\frac{bk^2\beta_2^2\lambda}{\lambda^2\sigma_1+\mu^2}=0,$$

$$2k^4c_1-2bk^2\alpha_1\alpha_2+\frac{\lambda}{\lambda^2\sigma_1+\mu^2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$-2bk^2\alpha_2\beta_2+6k^4d_2=0,$$

$$\left(\omega^2-k^2\right)c_2+8k^4c_2\lambda-bk^2\left(2\alpha_0\alpha_2+\alpha_1^2\right)-\frac{\lambda}{\lambda^2\sigma_1+\mu^2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)+\frac{bk^2\beta_2^2\lambda^2}{\lambda^2\sigma_1+\mu^2}=0,$$

$$-2bk^2\left(\alpha_1\beta_2+\alpha_2\beta_1\right)+k^4\left(2d_1-10c_2\mu\right)-\frac{2bk^2\beta_2^2\lambda\mu}{\lambda^2\sigma_1+\mu^2}=0,$$

$$\left(\omega^2-k^2\right)c_1+2k^4c_1\lambda-2bk^2\alpha_0\alpha_1+\frac{\lambda^2}{\lambda^2\sigma_1+\mu^2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$k^4\left(-3c_1\mu+5d_2\lambda\right)-2bk^2\left(\alpha_0\beta_2+\alpha_1\beta_1\right)+\left(\omega^2-k^2\right)d_2-\frac{2\lambda\mu}{\lambda^2\sigma_1+\mu^2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$k^4\lambda\left(d_1-4c_2\mu\right)-2bk^2\alpha_0\beta_1+\left(\omega^2-k^2\right)d_1+\frac{2\mu\lambda}{\lambda^2\sigma_1+\mu^2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)=0,$$

$$\left(\omega^2-k^2\right)c_0+2k^4\lambda^2c_2-bk^2\alpha_0^2-\frac{\lambda^2}{\lambda^2\sigma_1+\mu^2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)=0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu=0,\ \lambda=\frac{k^2-\omega^2}{2k^4},\alpha_0=0,\alpha_1=0,\alpha_2=0,\beta_1=0,\beta_2=\pm3\sqrt{\frac{2\sigma_1\left(\omega^2-k^2\right)}{a\,b}},\tag{23}$$

$$c_0=\frac{3\left(k^2-\omega^2\right)}{a\,k^2},c_1=0,c_2=\frac{6k^2}{a},d_1=0,d_2=0,\ p=-\frac{\omega}{2k},q=\frac{\omega^2-2k^2}{4k^2},$$

Where $\omega>k$.

From Equations 5, 16, 21, 22 and 23, we deduce the traveling wave solutions of SB-system (Equation 1) as follows:

$$u(\xi)=\left[\pm\frac{3\left(\omega^2-k^2\right)}{k^2}\sqrt{\frac{\sigma_1}{a\,b}}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})}{\left(A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right)^2}\right)\right]e^{i\eta},\tag{24}$$

$$v(\xi)=\frac{3\left(k^2-\omega^2\right)}{ak^2}+\frac{3\left(\omega^2-k^2\right)}{ak^2}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})}{A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})}\right)^2,\tag{25}$$

In particular, by setting $A_1=0$ and $A_2\neq0$ in Equations 24 and 25, we have the kink and bell shaped solitary solutions

$$u(\xi)=\left[\pm\frac{3\left(\omega^2-k^2\right)}{k^2\sqrt{a\,b}}i\,\mathrm{sech}(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\tanh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right]e^{i\eta},\tag{26}$$

$$v(\xi)=\left[-\frac{3\left(\omega^2-k^2\right)}{ak^2}\mathrm{sech}^2(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right],\tag{27}$$

while, if $A_1\neq0$ and $A_2=0$, then we have the anti-kink and anti-bell shaped solitary solutions

$$u(\xi)=\left[\pm\frac{3\left(\omega^2-k^2\right)}{k^2\sqrt{a\,b}}\mathrm{csch}(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\coth(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right]e^{i\eta},\tag{28}$$

$$v(\xi)=\left[\frac{3\left(\omega^2-k^2\right)}{ak^2}\mathrm{csch}^2(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right],\tag{29}$$

where $\xi=kx+\omega t,\ \eta=-\frac{\omega}{2k}x+\frac{\omega^2-2k^2}{4k^2}t.$

Result 2

Consider

$$\mu=0,\ \lambda=\frac{k^2-\omega^2}{k^4},\alpha_0=\pm\frac{2\left(k^2-\omega^2\right)}{k^2\sqrt{a\,b}},\alpha_1=0,\alpha_2=\pm\frac{3k^2}{\sqrt{a\,b}},\beta_1=0,$$
$$\beta_2=\pm 3\sqrt{\frac{\sigma_1\left(\omega^2-k^2\right)}{a\,b}},c_0=\frac{2\left(k^2-\omega^2\right)}{a\,k^2},c_1=0,c_2=\frac{3k^2}{a},d_1=0, \tag{30}$$
$$d_2=\frac{3\sqrt{\sigma_1\left(\omega^2-k^2\right)}}{a},\ p=-\frac{\omega}{2k},q=\frac{4k^2-5\omega^2}{4k^2},$$

where $\omega > k$.

In this result, we deduce the traveling wave solution of SB-system (Equation 1) as follows:

$$u(\xi)=\left[\pm\frac{2\left(k^2-\omega^2\right)}{k^2\sqrt{a\,b}}\pm\frac{3\left(\omega^2-k^2\right)}{k^2\sqrt{a\,b}}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}{A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}\right)^2\right. \tag{31}$$
$$\left.\pm\frac{3\left(\omega^2-k^2\right)}{k^2}\sqrt{\frac{\sigma_1}{a\,b}}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}{\left(A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right)^2}\right)\right]e^{i\eta},$$

$$v(\xi)=\frac{2\left(k^2-\omega^2\right)}{ak^2}+\frac{3\left(\omega^2-k^2\right)}{ak^2}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}{A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}\right)^2 \tag{32}$$
$$+\frac{3\left(\omega^2-k^2\right)\sqrt{\sigma_1}}{ak^2}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}{\left(A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right)^2}\right),$$

In particular, by setting $A_1 \neq 0$ and $A_2 = 0$ in Equations 31 and 32, we have the anti-kink and anti-bell shaped solitary solutions

$$u(\xi)=\frac{\left(\omega^2-k^2\right)}{k^2\sqrt{a\,b}}\left[\mp 2\pm 3\coth(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right. \tag{33}$$
$$\left.\times\left(\coth(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+\operatorname{csch}(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right)\right]e^{i\eta},$$

$$v(\xi)=\frac{\left(\omega^2-k^2\right)}{ak^2}\left[-2+3\coth(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right. \tag{34}$$
$$\left.\times\left(\coth(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+\operatorname{csch}(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right)\right],$$

where $\xi = kx+\omega t$, $\eta=-\frac{\omega}{2k}x+\frac{4k^2-5\omega^2}{4k^2}t$.

Case 2: Trigonometric function solution ($\lambda > 0$)

If $\lambda > 0$, substituting Equations 21 and 22 into Equations 19 and 20 and using Equations 4 and 8, the left-hand sides are converted into polynomial in ϕ and ψ. Setting each coefficient of this polynomial to 0, yields a system of algebraic equations in $\alpha_0,\alpha_1,\alpha_2,\beta_1,\ \beta_2,c_0,\ \ c_1,c_2,d_1$, d_2, μ, λ, ω, k, p and q as follows:

$$-a\alpha_2c_2+6k^2\alpha_2+\frac{a\beta_2d_2\lambda}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-2i(2kp+\omega)\alpha_2+2k^2\alpha_1-a\alpha_1c_2-a\alpha_2c_1-\frac{\lambda}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_2-a\beta_2d_1-6k^2\beta_2\mu\right)=0,$$

$$-a\alpha_2d_2+6k^2\beta_2-a\beta_2c_2=0,$$

$$-(p^2+q)\alpha_2-i(2kp+\omega)\alpha_1-a\alpha_0c_2-a\alpha_1c_1-a\alpha_2c_0+8k^2\alpha_2\lambda+\frac{\lambda^2a\beta_2d_2}{\mu^2-\lambda^2\sigma_2}$$
$$-\frac{\lambda}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_1+i(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)=0,$$

$$-2i(2kp+\omega)\beta_2-a\alpha_1d_2-a\alpha_2d_1-k^2\left(10\alpha_2\mu-2\beta_1\right)-a\beta_1c_2-a\beta_2c_1-\frac{2\lambda\mu a\beta_2d_2}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-(p^2+q)\alpha_1-2i(2kp+\omega)\alpha_2\lambda-a\alpha_0c_1-a\alpha_1c_0+2k^2\alpha_1\lambda$$
$$+\frac{\lambda^2}{\mu^2-\lambda^2\sigma_2}\left(a\beta_1d_2+a\beta_2d_1+6k^2\beta_2\mu\right)=0,$$

$$-a\alpha_0d_2-a\alpha_1d_1+k^2\left(-3\alpha_1\mu+5\beta_2\lambda\right)-a\beta_1c_1-a\beta_2c_0-(p^2+q)\beta_2+i(2kp+\omega)\left(-\beta_1+2\alpha_2\mu\right)$$
$$+\frac{2\lambda\mu}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_2-a\beta_2d_1-6k^2\beta_2\mu\right)=0,$$

$$k^2\lambda\left(\beta_1-4\alpha_2\mu\right)+i(2kp+\omega)\left(-\beta_2\lambda+\alpha_1\mu\right)-a\alpha_0d_1-(p^2+q)\beta_1-a\beta_1c_0$$
$$+\frac{2\lambda\mu}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_1+i(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)=0,$$

$$-i(2kp+\omega)\lambda\alpha_1+2k^2\alpha_2\lambda^2-a\alpha_0c_0-\frac{\lambda^2}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_1+i(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)$$
$$-(p^2+q)\alpha_0=0,$$

$$6k^4c_2-bk^2\alpha_2^2+\frac{bk^2\beta_2^2\lambda}{\mu^2-\lambda^2\sigma_2}=0,$$

$$2k^4c_1-2bk^2\alpha_1\alpha_2+\frac{\lambda}{\mu^2-\lambda^2\sigma_2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$-2bk^2\alpha_2\beta_2+6k^4d_2=0,$$

$$\left(\omega^2-k^2\right)c_2+8k^4c_2\lambda-bk^2\left(2\alpha_0\alpha_2+\alpha_1^2\right)-\frac{\lambda}{\mu^2-\lambda^2\sigma_2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)$$
$$+\frac{bk^2\beta_2^2\lambda^2}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-2bk^2\left(\alpha_1\beta_2+\alpha_2\beta_1\right)+k^4\left(2d_1-10c_2\mu\right)-\frac{2bk^2\beta_2^2\lambda\mu}{\mu^2-\lambda^2\sigma_2}=0,$$

$$\left(\omega^2-k^2\right)c_1+2k^4c_1\lambda-2bk^2\alpha_0\alpha_1+\frac{\lambda^2}{\mu^2-\lambda^2\sigma_2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$k^4\left(-3c_1\mu+5d_2\lambda\right)-2bk^2\left(\alpha_0\beta_2+\alpha_1\beta_1\right)+\left(\omega^2-k^2\right)d_2-\frac{2\lambda\mu}{\mu^2-\lambda^2\sigma_2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$k^4\lambda\left(d_1-4c_2\mu\right)-2bk^2\alpha_0\beta_1+\left(\omega^2-k^2\right)d_1+\frac{2\mu\lambda}{\mu^2-\lambda^2\sigma_2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)=0,$$

$$\left(\omega^2-k^2\right)c_0+2k^4\lambda^2c_2-bk^2\alpha_0^2-\frac{\lambda^2}{\mu^2-\lambda^2\sigma_2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)=0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu=0,\ \lambda=\frac{k^2-\omega^2}{2k^4},\alpha_0=0,\alpha_1=0,\alpha_2=0,\beta_1=0,\beta_2=\pm3\sqrt{\frac{2\sigma_2\left(k^2-\omega^2\right)}{a\,b}}, \quad (35)$$
$$c_0=\frac{3\left(k^2-\omega^2\right)}{a\,k^2},c_1=0,c_2=\frac{6k^2}{a},d_1=0,d_2=0,\ p=-\frac{\omega}{2k},q=\frac{\omega^2-2k^2}{4k^2},$$

Where $k>\omega$.

From Equations 7, 16, 21, 22 and 35, we deduce the traveling wave solutions of SB-system (Equation 1) as follows:

$$u(\xi)=\left[\pm\frac{3\left(k^2-\omega^2\right)}{k^2}\sqrt{\frac{\sigma_2}{a\,b}}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})}{\left(A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\right)^2}\right)\right]e^{i\eta}, \quad (36)$$

$$v(\xi)=\frac{3\left(k^2-\omega^2\right)}{ak^2}\left[1+\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})}{A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})}\right)^2\right], \quad (37)$$

In particular, by setting $A_1=0$ and $A_2\neq0$ in Equations 36 and 37, we have the periodic solutions

$$u(\xi)=\left[\mp\frac{3\left(k^2-\omega^2\right)}{k^2\sqrt{a\,b}}\sec(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\tan(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\right]e^{i\eta}, \quad (38)$$

$$v(\xi)=\frac{3\left(k^2-\omega^2\right)}{ak^2}\sec^2(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}}), \quad (39)$$

while, if $A_1\neq0$ and $A_2=0$, then we have the periodic solutions

$$u(\xi)=\left[\pm\frac{3\left(k^2-\omega^2\right)}{k^2\sqrt{a\,b}}\csc(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\cot(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\right]e^{i\eta}, \quad (40)$$

$$v(\xi)=\frac{3\left(k^2-\omega^2\right)}{ak^2}\csc^2(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}}), \quad (41)$$

where $\xi=kx+\omega t$, $\eta=-\frac{\omega}{2k}x+\frac{\omega^2-2k^2}{4k^2}t$.

Result 2

Consider

$$\mu=0,\ \lambda=\frac{k^2-\omega^2}{k^4},\alpha_0=\pm\frac{2\left(k^2-\omega^2\right)}{k^2\sqrt{a\,b}},\alpha_1=0,\alpha_2=\pm\frac{3k^2}{\sqrt{a\,b}},\beta_1=0,$$
$$\beta_2=\pm3\sqrt{\frac{\sigma_2\left(k^2-\omega^2\right)}{a\,b}},c_0=\frac{2\left(k^2-\omega^2\right)}{a\,k^2},c_1=0,c_2=\frac{3k^2}{a},d_1=0, \quad (42)$$
$$d_2=\frac{3\sqrt{\sigma_2\left(k^2-\omega^2\right)}}{a},\ p=-\frac{\omega}{2k},q=\frac{4k^2-5\omega^2}{4k^2},$$

Where $k>\omega$.

In this result, we deduce the traveling wave solution of SB-system (Equation 1) as follows:

$$u(\xi)=\left[\pm\frac{2\left(k^2-\omega^2\right)}{k^2\sqrt{a\,b}}\pm\frac{3\left(k^2-\omega^2\right)}{k^2\sqrt{a\,b}}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}{A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}\right)^2\right. \quad (43)$$
$$\left.\pm\frac{3\left(k^2-\omega^2\right)}{k^2}\sqrt{\frac{\sigma_2}{a\,b}}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}{\left(A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\right)^2}\right)\right]e^{i\eta},$$

$$v(\xi)=\frac{2\left(k^2-\omega^2\right)}{ak^2}+\frac{3\left(k^2-\omega^2\right)}{ak^2}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}{A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}\right)^2 \quad (44)$$
$$+\frac{3\left(k^2-\omega^2\right)\sqrt{\sigma_2}}{ak^2}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}{\left(A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\right)^2}\right),$$

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in Equations 43 and 44, we have the periodic solutions

$$u(\xi) = \pm \frac{\left(k^2 - \omega^2\right)}{k^2 \sqrt{a\, b}} \left[2 + 3\tan(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) \times \left(\tan(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) - \sec(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) \right) \right] e^{i\eta}, \quad (45)$$

$$v(\xi) = \frac{\left(k^2 - \omega^2\right)}{ak^2} \left[2 + 3\tan(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) \left(\tan(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) - \sec(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) \right) \right], \quad (46)$$

while, if $A_1 \neq 0$ and $A_2 = 0$, then we have the periodic solutions

$$u(\xi) = \pm \frac{\left(k^2 - \omega^2\right)}{k^2 \sqrt{a\, b}} \left[2 + 3\cot(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) \times \left(\cot(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) + \csc(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) \right) \right] e^{i\eta}, \quad (47)$$

$$v(\xi) = \frac{\left(k^2 - \omega^2\right)}{ak^2} \left[2 + 3\cot(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) \left(\cot(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) + \csc(\xi \sqrt{\frac{k^2 - \omega^2}{k^4}}) \right) \right], \quad (48)$$

Where $\xi = kx + \omega t$, $\eta = -\frac{\omega}{2k} x + \frac{4k^2 - 5\omega^2}{4k^2} t.$

Case 3: Rational function solutions ($\lambda = 0$)

If $\lambda = 0$, substituting Equations 21 and 22 into Equations 19 and 20 and using Equations 4 and 10, the left-hand sides are converted into polynomial in ϕ and ψ. Setting each coefficient of this polynomial to 0, yields a system of algebraic equations in $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, c_0, c_1, c_2, d_1, d_2, \mu, \omega, k$, p and q as follows:

$$-a\alpha_2 c_2 + 6k^2 \alpha_2 - \frac{a\beta_2 d_2}{A_1^2 - 2\mu A_2} = 0,$$

$$-2i(2kp + \omega)\alpha_2 + 2k^2 \alpha_1 - a\alpha_1 c_2 - a\alpha_2 c_1 - \frac{1}{A_1^2 - 2\mu A_2}\left(a\beta_1 d_2 + a\beta_2 d_1 + 6k^2 \beta_2 \mu\right) = 0,$$

$$-a\alpha_2 d_2 + 6k^2 \beta_2 - a\beta_2 c_2 = 0,$$

$$-(p^2 + q)\alpha_2 - i(2kp + \omega)\alpha_1 - a\alpha_0 c_2 - a\alpha_1 c_1 - a\alpha_2 c_0 + \frac{1}{A_1^2 - 2\mu A_2}\left(-a\beta_1 d_1 + i(2kp + \omega)\beta_2 \mu + k^2\left(2\alpha_2 \mu^2 - \beta_1 \mu\right)\right) = 0,$$

$$-2i(2kp + \omega)\beta_2 - a\alpha_1 d_2 - a\alpha_2 d_1 - k^2\left(10\alpha_2 \mu - 2\beta_1\right) - a\beta_1 c_2 - a\beta_2 c_1 + \frac{2\mu a\beta_2 d_2}{A_1^2 - 2\mu A_2} = 0,$$

$$-(p^2 + q)\alpha_1 - a\alpha_0 c_1 - a\alpha_1 c_0 = 0,$$

$$-a\alpha_0 d_2 - a\alpha_1 d_1 - 3\alpha_1 \mu k^2 - a\beta_1 c_1 - a\beta_2 c_0 - (p^2 + q)\beta_2 + i(2kp + \omega)\left(-\beta_1 + 2\alpha_2 \mu\right) + \frac{2\mu}{A_1^2 - 2\mu A_2}\left(a\beta_1 d_2 + a\beta_2 d_1 + 6k^2 \beta_2 \mu\right) = 0,$$

$$i(2kp + \omega)\alpha_1 \mu - a\alpha_0 d_1 - (p^2 + q)\beta_1 - a\beta_1 c_0 - \frac{2\mu}{A_1^2 - 2\mu A_2}\left(-a\beta_1 d_1 + i(2kp + \omega)\beta_2 \mu + k^2\left(2\alpha_2 \mu^2 - \beta_1 \mu\right)\right) = 0,$$

$$-a\alpha_0 c_0 - (p^2 + q)\alpha_0 = 0,$$

$$6k^4 c_2 - bk^2 \alpha_2^2 - \frac{bk^2 \beta_2^2}{A_1^2 - 2\mu A_2} = 0,$$

$$2k^4 c_1 - 2bk^2 \alpha_1 \alpha_2 - \frac{1}{A_1^2 - 2\mu A_2}\left(6k^4 d_2 \mu + 2bk^2 \beta_1 \beta_2\right) = 0,$$

$$-2bk^2 \alpha_2 \beta_2 + 6k^4 d_2 = 0,$$

$$\left(\omega^2 - k^2\right)c_2 - bk^2\left(2\alpha_0 \alpha_2 + \alpha_1^2\right) + \frac{1}{A_1^2 - 2\mu A_2}\left(k^4\left(2c_2 \mu^2 - d_1 \mu\right) - bk^2 \beta_1^2\right) = 0,$$

$$-2bk^2\left(\alpha_1 \beta_2 + \alpha_2 \beta_1\right) + k^4\left(2d_1 - 10c_2 \mu\right) + \frac{2bk^2 \beta_2^2 \mu}{A_1^2 - 2\mu A_2} = 0,$$

$$\left(\omega^2 - k^2\right)c_1 - 2bk^2 \alpha_0 \alpha_1 = 0,$$

$$-3c_1 \mu k^4 - 2bk^2\left(\alpha_0 \beta_2 + \alpha_1 \beta_1\right) + \left(\omega^2 - k^2\right)d_2 + \frac{2\mu}{A_1^2 - 2\mu A_2}\left(6k^4 d_2 \mu + 2bk^2 \beta_1 \beta_2\right) = 0,$$

$$-2bk^2 \alpha_0 \beta_1 + \left(\omega^2 - k^2\right)d_1 - \frac{2\mu}{A_1^2 - 2\mu A_2}\left(k^4\left(2c_2 \mu^2 - d_1 \mu\right) - bk^2 \beta_1^2\right) = 0,$$

$$\left(\omega^2 - k^2\right)c_0 - bk^2 \alpha_0^2 = 0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu = 0,, \alpha_0 = 0, \alpha_1 = 0, \alpha_2 = \pm \frac{6k^2}{\sqrt{a\, b}}, \beta_1 = 0, \beta_2 = 0, c_0 = -\frac{(4q + 1)}{4a}, \quad (49)$$
$$c_1 = 0, c_2 = \frac{6k^2}{a}, d_1 = 0, d_2 = 0, p = -\frac{1}{2}, q = q, \omega = k.$$

From Equations 9, 16, 21, 22 and 49, we deduce the traveling wave solutions of SB-system (Equation 1) as follows:

$$u(\xi) = \pm \frac{6k^2}{\sqrt{ab}} \left(\frac{A_1}{A_1\xi + A_2} \right)^2 e^{i\eta}, \tag{50}$$

$$v(\xi) = -\frac{(4q+1)}{4a} + \frac{6k^2}{a} \left(\frac{A_1}{A_1\xi + A_2} \right)^2, \tag{51}$$

Where $\xi = kx + \omega t$, $\eta = -\frac{1}{2}x + qt$.

Example 2

*The (2+1)-dimensional HNLS equation (*Equation 2*)*

Here, we study the (2+1)-dimensional HNLS Equation 2. To this end, we assume that the solution of Equation 2 can be written as:

$$u(x,y,t) = W(\xi)e^{i\eta}, \quad \xi = x + ay - ct, \eta = mx + ny + \omega t, \tag{52}$$

where $W(\xi)$ is a real function of ξ and a, c, m, n, ω are constants to be determined. Substituting Equation 52 into Equation 2, we obtain

$$(c^2-1)W'' - \left[\omega^2 - 2n - (a+\omega c)^2\right]W - 2W^3 = 0, \tag{53}$$

Where $c^2 \neq 1$.

By balancing between W'' with W^3 in (53) we get $N + 2 = 3N \Rightarrow N = 1$. Consequently, Equation 53 has the formal solution:

$$W(\xi) = \alpha_0 + \alpha_1\phi(\xi) + \beta_1\psi(\xi), \tag{54}$$

Where α_0, α_1 and β_1 are constants to be determined later satisfying $\alpha_1^2 + \beta_1^2 \neq 0$. There are three cases to be discussed as follows:

Case 1: Hyperbolic function solutions ($\lambda < 0$)

If $\lambda < 0$, substituting Equation 54 into Equation 53 and using Equations 4 and 6, the left-hand side of Equation 53 becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be 0, yields a system of algebraic equations in $\alpha_0, \alpha_1, \beta_1, a, c, m, n, \omega$, μ and λ as follows:

$$2\left(c^2-1\right)\alpha_1 - 2\alpha_1^3 - \frac{6\alpha_1\beta_1^2\lambda}{\lambda^2\sigma_1 + \mu^2} = 0,$$

$$-6\alpha_0\alpha_1^2 + \frac{4\beta_1^3\lambda^2\mu}{\left(\lambda^2\sigma_1 + \mu^2\right)^2} + \frac{\lambda}{\lambda^2\sigma_1 + \mu^2}\left(\left(c^2-1\right)\beta_1\mu + 6\alpha_0\beta_1^2\right) = 0,$$

$$-6\alpha_1^2\beta_1 + 2\left(c^2-1\right)\beta_1 + \frac{2\beta_1^3\lambda}{\lambda^2\sigma_1 + \mu^2} = 0,$$

$$-(\omega^2 - 2n - (a+\omega c)^2)\alpha_1 - 6\alpha_0^2\alpha_1 + 2\left(c^2-1\right)\alpha_1\lambda + \frac{6\alpha_1\beta_1^2\lambda^2}{\lambda^2\sigma_1 + \mu^2} = 0,$$

$$-3\left(c^2-1\right)\alpha_1\mu - 12\alpha_0\alpha_1\beta_1 - \frac{12\alpha_1\beta_1^2\lambda\mu}{\lambda^2\sigma_1 + \mu^2} = 0,$$

$$-(\omega^2 - 2n - (a+\omega c)^2)\beta_1 - 6\alpha_0^2\beta_1 + \left(c^2-1\right)\beta_1\lambda - \frac{8\beta_1^3\lambda^2\mu^2}{\left(\lambda^2\sigma_1 + \mu^2\right)^2} + \frac{2\beta_1^3\lambda^2}{\lambda^2\sigma_1 + \mu^2}$$
$$- \frac{2\lambda\mu}{\lambda^2\sigma_1 + \mu^2}\left(\left(c^2-1\right)\beta_1\mu + 6\alpha_0\beta_1^2\right) = 0,$$

$$-(\omega^2 - 2n - (a+\omega c)^2)\alpha_0 - 2\alpha_0^3 + \frac{4\beta_1^3\lambda^3\mu}{\left(\lambda^2\sigma_1 + \mu^2\right)^2} + \frac{\lambda^2}{\lambda^2\sigma_1 + \mu^2}\left(\left(c^2-1\right)\beta_1\mu + 6\alpha_0\beta_1^2\right) = 0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu = 0, \lambda = \lambda, \alpha_0 = 0, \alpha_1 = \pm\sqrt{c^2-1}, \beta_1 = 0, a = a, c = c, \omega = \omega,$$
$$n = (\frac{1}{2}\omega^2 + \lambda)(1-c^2) - a(\frac{1}{2}a + c\omega), \tag{55}$$

Where $c^2 > 1$.

From Equations 5, 52, 54 and 55, we deduce the traveling wave solution of Equation 2 as follows:

$$u(x,y,t) = \pm\sqrt{-\lambda(c^2-1)}\left(\frac{A_1\cosh(\xi\sqrt{-\lambda}) + A_2\sinh(\xi\sqrt{-\lambda})}{A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda})}\right)e^{i\eta}, \tag{56}$$

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in Equation 56, we have the kink shaped solitary solution

$$u(x,y,t) = \pm\sqrt{-\lambda(c^2-1)}\tanh(\xi\sqrt{-\lambda})e^{i\eta}, \tag{57}$$

While, if $A_1 \neq 0$ and $A_2 = 0$, then we have the anti-kink shaped solitary solution

$$u(x,y,t) = \pm\sqrt{-\lambda(c^2-1)}\coth(\xi\sqrt{-\lambda})e^{i\eta}, \tag{58}$$

Where

$$\xi = x + ay - ct,\ \eta = -(a+\omega c)x + \left((\frac{1}{2}\omega^2+\lambda)(1-c^2) - a(\frac{1}{2}a + c\omega)\right)y + \omega t.$$

Result 2

Consider

$$\mu=\mu,\ \lambda=\lambda,\alpha_0=0,\alpha_1=\pm\frac{\sqrt{c^2-1}}{2},\beta_1=\pm\sqrt{-\frac{(c^2-1)(\lambda^2\sigma_1+\mu^2)}{4\lambda}},a=a,$$
$$c=c,\omega=\omega,n=(1-c^2)\left(\frac{1}{2}\omega^2+\frac{1}{4}\lambda\right)-a(\frac{1}{2}a+c\omega), \quad (59)$$

where $c^2 > 1$.

In this result, we deduce the traveling wave solution of Equation 2 as follows:

$$u(x,y,t)=\left[\pm\frac{\sqrt{-\lambda(c^2-1)}}{2}\left(\frac{A_1\cosh(\xi\sqrt{-\lambda})+A_2\sinh(\xi\sqrt{-\lambda})}{A_1\sinh(\xi\sqrt{-\lambda})+A_2\cosh(\xi\sqrt{-\lambda})+\frac{\mu}{\lambda}}\right)\right. \quad (60)$$
$$\left.\pm\sqrt{-\frac{(c^2-1)(\lambda^2\sigma_1+\mu^2)}{4\lambda}}\left(\frac{1}{A_1\sinh(\xi\sqrt{-\lambda})+A_2\cosh(\xi\sqrt{-\lambda})+\frac{\mu}{\lambda}}\right)\right]e^{i\eta},$$

In particular, by setting $A_1 \neq 0$, $A_2 = 0$ and $\mu = 0$ in Equation 60, we have the anti-kink and anti-bell shaped solitary solution

$$u(x,y,t)=\pm\frac{\sqrt{-\lambda(c^2-1)}}{2}\left(\coth(\xi\sqrt{-\lambda})+\text{csch}(\xi\sqrt{-\lambda})\right)e^{i\eta}, \quad (61)$$

Where

$$\xi = x + ay - ct,\ \eta = -(a+\omega c)x + \left((1-c^2)\left(\frac{1}{2}\omega^2+\frac{1}{4}\lambda\right) - a(\frac{1}{2}a + c\omega)\right)y + \omega t.$$

Case 2: Trigonometric function solution ($\lambda > 0$)

If $\lambda > 0$, substituting Equation 54 into Equation 53 and using Equations 4 and 8, the left-hand side of Equation 53 becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be 0, yields a system of algebraic equations in $\alpha_0, \alpha_1, \beta_1, a, c, m, n, \omega, \mu$ and λ as follows:

$$2(c^2-1)\alpha_1 - 2\alpha_1^3 + \frac{6\alpha_1\beta_1^2\lambda}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-6\alpha_0\alpha_1^2 + \frac{4\beta_1^3\lambda^2\mu}{(\mu^2-\lambda^2\sigma_2)^2} + \frac{\lambda}{\mu^2-\lambda^2\sigma_2}\left((c^2-1)\beta_1\mu + 6\alpha_0\beta_1^2\right)=0,$$

$$-6\alpha_1^2\beta_1 + 2(c^2-1)\beta_1 + \frac{2\beta_1^3\lambda}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-(\omega^2-2n-(a+\omega c)^2)\alpha_1 - 6\alpha_0^2\alpha_1 + 2(c^2-1)\alpha_1\lambda + \frac{6\alpha_1\beta_1^2\lambda^2}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-3(c^2-1)\alpha_1\mu - 12\alpha_0\alpha_1\beta_1 - \frac{12\alpha_1\beta_1^2\lambda\mu}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-(\omega^2-2n-(a+\omega c)^2)\beta_1 - 6\alpha_0^2\beta_1 + (c^2-1)\beta_1\lambda - \frac{8\beta_1^3\lambda^2\mu^2}{(\mu^2-\lambda^2\sigma_2)^2} + \frac{2\beta_1^3\lambda^2}{\mu^2-\lambda^2\sigma_2}$$
$$-\frac{2\lambda\mu}{\mu^2-\lambda^2\sigma_2}\left((c^2-1)\beta_1\mu + 6\alpha_0\beta_1^2\right)=0,$$

$$-(\omega^2-2n-(a+\omega c)^2)\alpha_0 - 2\alpha_0^3 + \frac{4\beta_1^3\lambda^3\mu}{(\mu^2-\lambda^2\sigma_2)^2} + \frac{\lambda^2}{\mu^2-\lambda^2\sigma_2}\left((c^2-1)\beta_1\mu + 6\alpha_0\beta_1^2\right)=0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu=0,\ \lambda=\lambda,\alpha_0=0,\alpha_1=\pm\sqrt{c^2-1},\beta_1=0,a=a,c=c,\omega=\omega,$$
$$n=(\frac{1}{2}\omega^2+\lambda)(1-c^2)-a(\frac{1}{2}a+c\omega), \quad (62)$$

Where $c^2 > 1$.

From Equations 7, 52, 54 and 62, we deduce the traveling wave solutions of Equations 2 as follows:

$$u(x,y,t)=\pm\sqrt{\lambda(c^2-1)}\left(\frac{A_1\cos(\xi\sqrt{\lambda})-A_2\sin(\xi\sqrt{\lambda})}{A_1\sin(\xi\sqrt{\lambda})+A_2\cos(\xi\sqrt{\lambda})}\right)e^{i\eta}, \quad (63)$$

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in Equation 63, we have the periodic solution

$$u(x,y,t)=\mp\sqrt{\lambda(c^2-1)}\tan(\xi\sqrt{\lambda})e^{i\eta}, \quad (64)$$

while, if $A_1 \neq 0$ and $A_2 = 0$, then we have the periodic solution

$$u(x,y,t)=\pm\sqrt{\lambda(c^2-1)}\cot(\xi\sqrt{\lambda})e^{i\eta}, \quad (65)$$

where

$$\xi = x + ay - ct,\ \eta = -(a+\omega c)x + \left((\frac{1}{2}\omega^2+\lambda)(1-c^2) - a(\frac{1}{2}a+c\omega)\right)y + \omega t.$$

Result 2

Consider

$$\mu=\mu,\ \lambda=\lambda,\alpha_0=0,\alpha_1=\pm\frac{\sqrt{c^2-1}}{2},\beta_1=\pm\sqrt{\frac{(c^2-1)(\lambda^2\sigma_2-\mu^2)}{4\lambda}},a=a, \quad (66)$$
$$c=c,\omega=\omega,n=(1-c^2)\left(\frac{1}{2}\omega^2+\frac{1}{4}\lambda\right)-a(\frac{1}{2}a+c\omega),$$

Where $c^2>1$.

In this result, we deduce the traveling wave solution of Equation 2 as follows:

$$u(x,y,t)=\left[\pm\frac{\sqrt{\lambda(c^2-1)}}{2}\left(\frac{A_1\cos(\xi\sqrt{\lambda})-A_2\sin(\xi\sqrt{\lambda})}{A_1\sin(\xi\sqrt{\lambda})+A_2\cos(\xi\sqrt{\lambda})+\frac{\mu}{\lambda}}\right)\right. \quad (67)$$
$$\left.\pm\sqrt{\frac{(c^2-1)(\lambda^2\sigma_2-\mu^2)}{4\lambda}}\left(\frac{1}{A_1\sin(\xi\sqrt{\lambda})+A_2\cos(\xi\sqrt{\lambda})+\frac{\mu}{\lambda}}\right)\right]e^{i\eta},$$

In particular, by setting $A_1=0$, $A_2\neq 0$ and $\mu=0$ in Equation 67, we have the periodic solution

$$u(x,y,t)=\pm\frac{\sqrt{\lambda(c^2-1)}}{2}\left(-\tan(\xi\sqrt{\lambda})+\sec(\xi\sqrt{\lambda})\right)e^{i\eta}, \quad (68)$$

while, if $A_1\neq 0$, $A_2=0$ and $\mu=0$, then we have the periodic solution

$$u(x,y,t)=\pm\frac{\sqrt{\lambda(c^2-1)}}{2}\left(\cot(\xi\sqrt{\lambda})+\csc(\xi\sqrt{\lambda})\right)e^{i\eta}, \quad (69)$$

Where

$$\xi = x + ay - ct,\ \eta = -(a+\omega c)x + \left((1-c^2)\left(\frac{1}{2}\omega^2+\frac{1}{4}\lambda\right) - a(\frac{1}{2}a+c\omega)\right)y + \omega t.$$

Case 3: Rational function solutions ($\lambda=0$)

If $\lambda=0$, substituting Equation 54 into Equation 53 and using Equations 4 and 10, the left-hand side of Equation 53 becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be 0, yields a system of algebraic equations in $\alpha_0,\alpha_1,\beta_1,a,c,m,n,\omega$ and μ as follows:

$$2(c^2-1)\alpha_1-2\alpha_1^3-\frac{6\alpha_1\beta_1^2}{A_1^2-2\mu A_2}=0,$$

$$-6\alpha_0\alpha_1^2+\frac{4\beta_1^3\mu}{(A_1^2-2\mu A_2)^2}-\frac{1}{A_1^2-2\mu A_2}\left((c^2-1)\beta_1\mu+6\alpha_0\beta_1^2\right)=0,$$

$$-6\alpha_1^2\beta_1+2(c^2-1)\beta_1-\frac{2\beta_1^3}{A_1^2-2\mu A_2}=0,$$

$$-(\omega^2-2n-(a+\omega c)^2)\alpha_1-6\alpha_0^2\alpha_1=0,$$

$$-3(c^2-1)\alpha_1\mu-12\alpha_0\alpha_1\beta_1+\frac{12\alpha_1\beta_1^2\mu}{A_1^2-2\mu A_2}=0,$$

$$-(\omega^2-2n-(a+\omega c)^2)\beta_1-6\alpha_0^2\beta_1-\frac{8\beta_1^3\mu^2}{(A_1^2-2\mu A_2)^2}$$
$$+\frac{2\mu}{A_1^2-2\mu A_2}\left((c^2-1)\beta_1\mu+6\alpha_0\beta_1^2\right)=0,$$

$$-(\omega^2-2n-(a+\omega c)^2)\alpha_0-2\alpha_0^3=0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu=0,\alpha_0=0,\alpha_1=0,\beta_1=\pm A_1\sqrt{c^2-1},a=a,c=c,\omega=\omega, \quad (70)$$
$$n=\frac{1}{2}\omega^2(1-c^2)-a(\frac{1}{2}a+c\omega),$$

where $c^2>1$.

From Equations 9, 52, 54 and 70, we deduce the traveling wave solutions of Equation 2 as follows:

$$u(x,y,t)=\pm\sqrt{c^2-1}\left(\frac{A_1}{A_1\xi+A_2}\right)e^{i\eta}, \quad (71)$$

Where

$$\xi = x + ay - ct,\ \eta = -(a+\omega c)x + \left(\frac{1}{2}\omega^2(1-c^2) - a(\frac{1}{2}a+c\omega)\right)y + \omega t.$$

Result 2

Consider

$$\mu=\mu,\alpha_0=0,\alpha_1=\pm\frac{\sqrt{c^2-1}}{2},\beta_1=\pm\frac{\sqrt{(c^2-1)(A_1^2-2\mu A_2)}}{2},a=a, \quad (72)$$
$$c=c,\omega=\omega,n=\frac{1}{2}\omega^2(1-c^2)-a(\frac{1}{2}a+c\omega),$$

Where $c^2>1, A_1^2>2\mu A_2$.

In this result, we deduce the traveling wave solution of Equation 2 as follows:

$$u(x,y,t)=\left[\frac{\pm\sqrt{c^2-1}}{2}\left(\frac{\mu\xi+A_1}{\frac{\mu}{2}\xi^2+A_1\xi+A_2}\right)\pm\frac{\sqrt{(c^2-1)(A_1^2-2\mu A_2)}}{2}\times\left(\frac{1}{\frac{\mu}{2}\xi^2+A_1\xi+A_2}\right)\right]e^{i\eta}, \quad (73)$$

where

$$\xi=x+ay-ct,\ \eta=-(a+\omega c)x+\left(\frac{1}{2}\omega^2(1-c^2)-a(\frac{1}{2}a+c\omega)\right)y+\omega t.$$

PHYSICAL EXPLANATIONS OF SOME OBTAINED SOLUTIONS

The obtained solutions for the two equations (1) and (2) include the kink, anti-kink soliton solutions, bell and anti-bell soliton solutions as well as periodic and rational solutions. The graphical representations of some of these solutions are plotted by taking suitable values of involved unknown parameters to visualize the mechanism of the original equations (Figures 1 to 6).

CONCLUSIONS

The two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method is used in this article to obtain new exact solutions of two nonlinear PDEs namely, the (1+1)-dimensional nonlinear Schrödinger-Boussinesq system and the (2+1)-dimensional HNLS equation. These exact solutions are presented in terms of the hyperbolic, trigonometric and rational functions. As the two parameters A_1 and A_2 takes special values, we obtain the solitary wave solutions. From Equations 3 and 14, we can deduce that the two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method reduces to the $\left(\frac{G'}{G}\right)$-expansion method. So the two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method is an extension of the $\left(\frac{G'}{G}\right)$-expansion method. The used method in this paper is more effective and more general than the $\left(\frac{G'}{G}\right)$-expansion method because it gives exact solutions in more general forms. In summary, the advantage of the two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method over the $\left(\frac{G'}{G}\right)$-expansion method is that the solutions obtained by using

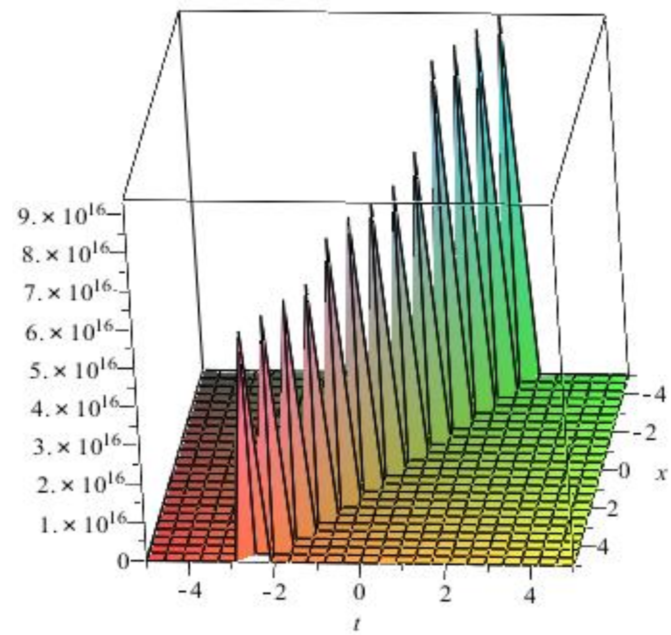

Figure 1. The plot of $U(x,t)$ of Equation 33 when $k=1,\omega=2,a=1,b=1.$

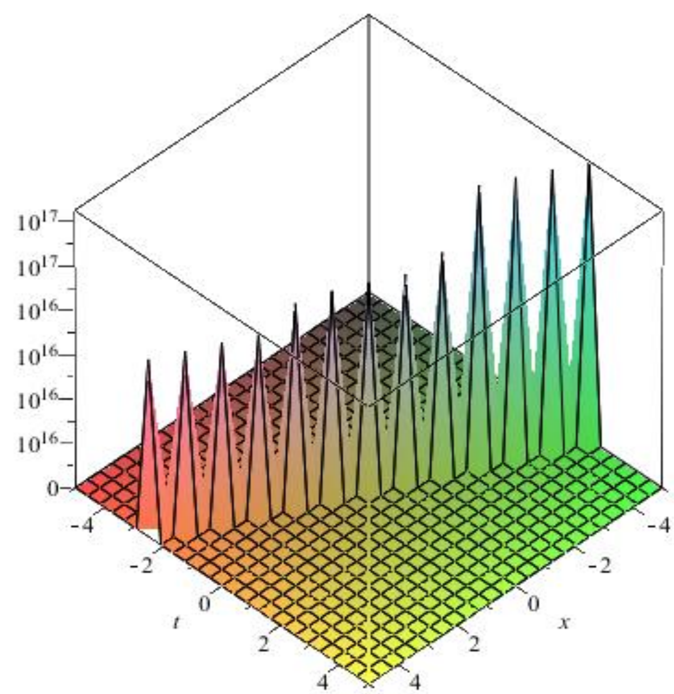

Figure 2. The plot of $v(x,t)$ of Equation 34 when $k=2,\omega=3,a=3.$

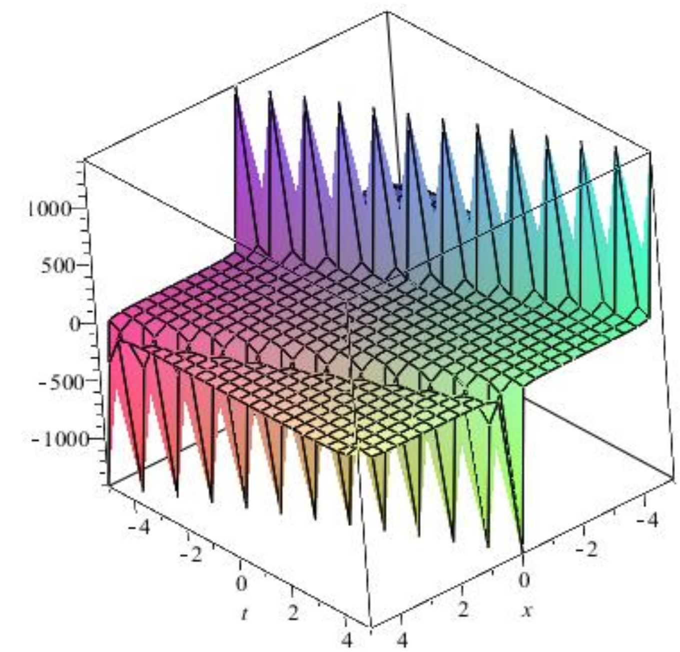

Figure 3. The plot of $U(x,t)$ of Equation 38 when $k=2, \omega=1, a=1, b=1$.

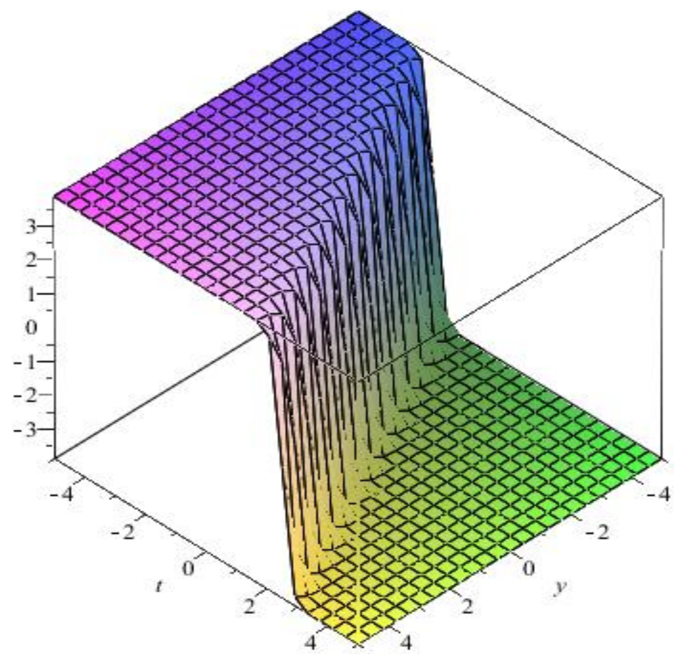

Figure 5. The plot of $W(0,y,t)$ of Equation 57 when $\lambda=-1, c=4, a=2$.

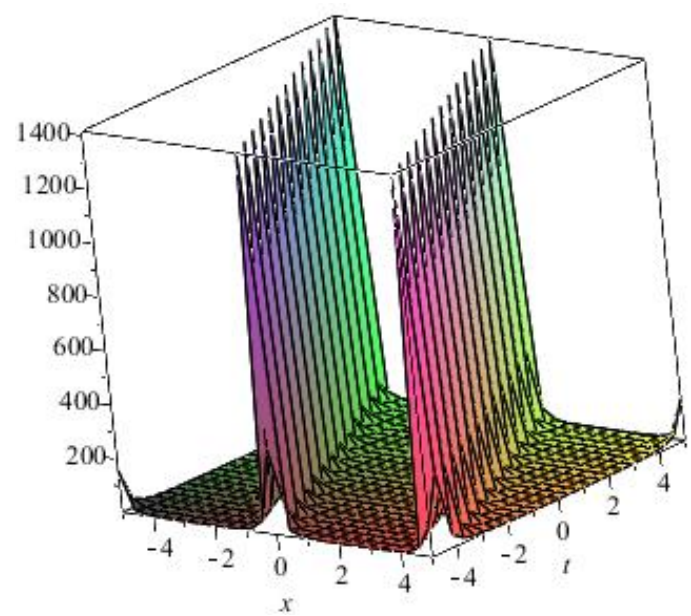

Figure 4. The plot of $v(x,t)$ of Equation 39 when $k=2, \omega=1, a=1$.

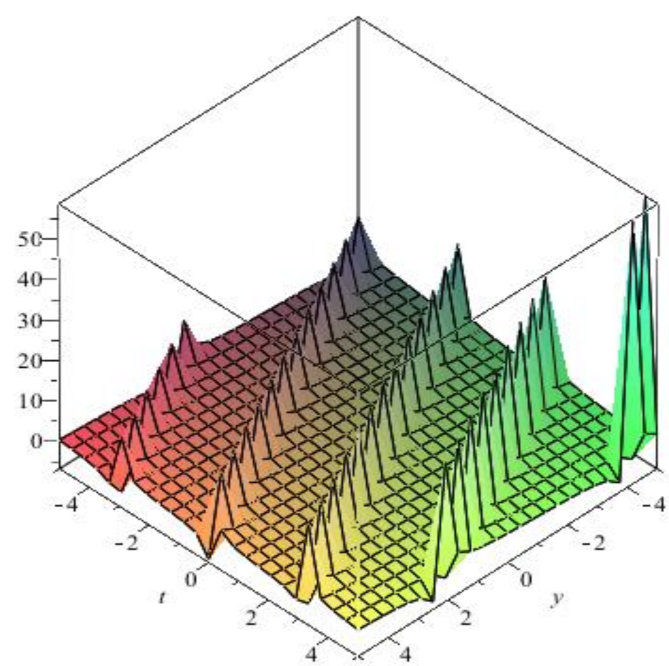

Figure 6. The plot of $W(0,y,t)$ of Equation 68 when $\lambda=1, c=2, a=1$.

the first method recover the solutions obtained by using the second one. On comparing our results obtained in this article with the well-know results obtained in Kilicman and Abazari (2012) and Fen (2012), we conclude that our results are new and not published elsewhere.

Finally, all solutions obtained in this article have been checked with the Maple by putting them back into the original equations.

Conflict of Interest

The authors have not declared any conflict of interest.

ACKNOWLEDGEMENT

The authors wish to thank the referees for their comments.

REFERENCES

Ablowitz MJ, Clarkson PA (1991). Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, New York, NY,USA,. http://dx.doi.org/10.1017/CBO9780511623998 PMid:9905818

Abdou MA (2010). The extended tanh- method and its applications for solving nonlinear physical models, Appl. Math. Comput. 190 988-996. http://dx.doi.org/10.1016/j.amc.2007.01.070

Aslan I (2010). A note on the $\left(\frac{G'}{G}\right)$expansion method again, Appl. Math. Comput., 217:937-938.http://dx.doi.org/10.1016/j.amc.2010.05.097

Ayhan B, Bekir A (2012). The $\left(\frac{G'}{G}\right)$ expansion method for the nonlinear lattice equations, Commu. Nonlinear Sci. Numer. Simul. 17:3490-3498. http://dx.doi.org/10.1016/j.cnsns.2012.01.009

Bekir A (2008). Application of the $\left(\frac{G'}{G}\right)$ expansion method for nonlinear evolution equations, Phys. Lett. A, 372:3400-3406. http://dx.doi.org/10.1016/j.physleta.2008.01.057

Bekir A (2009). The exp-function for Ostrovsky equation. Int. J. Nonlinear Sci. Num. Simul. 10 735-739. http://dx.doi.org/10.1515/IJNSNS.2009.10.6.735

Bekir A (2010). Application of the exp-function method for nonlinear differential-difference equations. Appl. Math. Comput. 215:4049-4053. http://dx.doi.org/10.1016/j.amc.2009.12.003

Chen Y, Wang Q (2005). Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic function solutions to (1+1) -dimensional dispersive long wave equation, Chaos, Solitons and Fractals, 24:745-757. http://dx.doi.org/10.1016/j.chaos.2004.09.014

Fen XY (2012). Bifurcations of exact traveling wave solutions for (2+1)-dimensional HNLS equation. Commun. Theor. Phys. 57:68-70. http://dx.doi.org/10.1088/0253-6102/57/1/11

Fan EG (2000). Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A, 277:212-218. http://dx.doi.org/10.1016/S0375-9601(00)00725-8

Gorz SP, Haelterman M (2008). Ultrafast transverse undulation of self-trapped laser beams, Optics Express, 16:16935. http://dx.doi.org/10.1364/OE.16.016935 PMid:18852801

He JH, Wu XH (2006). Exp-function method for nonlinear wave equations, Chaos, Solitons and Fractals, 30:700-708. http://dx.doi.org/10.1016/j.chaos.2006.03.020

Hirota R (1971). Exact solutions of the KdV equation for multiple collisions of solutions, Phys. Rev. Lett. 27:1192-1194. http://dx.doi.org/10.1103/PhysRevLett.27.1192

Kudryashov NA (1988). Exact soliton solutions of a generalized evolution equation of wave dynamics, J. Appl. Math. Mech. 52:361-365. http://dx.doi.org/10.1016/0021-8928(88)90090-1

Kudryashov NA (1990). Exact solutions of the generalized Kuramoto-Sivashinsky equation, Phys. Lett. A, 147:287-291. http://dx.doi.org/10.1016/0375-9601(90)90449-X

Kudryashov NA (1991). On types of nonlinear non-integrable equations with exact solutions, Phys. Lett. A, 155:269-275. http://dx.doi.org/10.1016/0375-9601(91)90481-M

Kudryashov NA (2010a). A note on the $\left(\frac{G'}{G}\right)$ expansion method. Appl. Math. Comput. 217:1755-1758. http://dx.doi.org/10.1016/j.amc.2010.03.071

Kudryashov NA (2010b). Meromorphic solutions of nonlinear ordinary differential equations. Comm. Nonlinear. Sci. Numer. Simul. 15:2778-2790. http://dx.doi.org/10.1016/j.cnsns.2009.11.013

Kilicman A, Abazari R (2012). Travelling wave solutions of the Schrödinger-Boussinesq system. Abst. Appl. Anal. Article ID 198398, 11 Pages.

Liu S, Fu Z, Liu S, Zhao Q (2001). Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys. Lett. A, 289:69-74. http://dx.doi.org/10.1016/S0375-9601(01)00580-1

Lu D (2005). Jacobi elliptic function solutions for two variant Boussinesq equations, Chaos, Solitons and Fractals, 24:1373-1385. http://dx.doi.org/10.1016/j.chaos.2004.09.085

Li B (2007). A generalized sub-equation expansion method and its application to the nonlinear Schrödinger equation in inhomogeneous optical fiber media. Int. J. Modern Phys. C, 18:1187–1201. http://dx.doi.org/10.1142/S0129183107011224

Li LX Li QE, Wang LM (2010). The $\left(\frac{G'}{G},\frac{1}{G}\right)$ expansion method and its application to travelling wave solutions of the Zakharov equations, Appl. Math. J. Chinese. Uni. 25:454-462.

Ma WX, Fuchssteiner B (1996). Explicit and exact solutions of KPP equation. Int. J. Nonlinear Mech. 31:329-338. http://dx.doi.org/10.1016/0020-7462(95)00064-X

Ma WX (2011). Generalized bilinear differential equations. Stud. Nonlinear Sci. 2:140-144.

Ma WX (2013). Bilinear equations and resonant solutions characterized by bell polynomial, reports on Math. Phys. 72:41-56.

Ma WX Lee JH (2009). A transformed rational function method and exact solutions to the (3+1)-dimensional Jimbo Miwa equation, Chaos, Solitons and Fractals, 42:1356-1363. http://dx.doi.org/10.1016/j.chaos.2009.03.043

Ma WX, Zhu Z (2012) Solving the (3+1)-dimensional generalized KP and BKP by the multiple exp-function method, Appl. Math. Comput. 218:11871-11879.http://dx.doi.org/10.1016/j.amc.2012.05.049

Makhankov V (1974). On stationary solutions of Schrödinger equation with a self- consistent potential satisfying Boussinesqs equations, Phys. Lett. A, 50:42–44. http://dx.doi.org/10.1016/0375-9601(74)90344-2

Miura MR (1978). Backlund Transformation, Springer, Berlin, Germany. http://dx.doi.org/10.1063/1.525721

Rogers C, Shadwick WF (1982). Backlund Transformations and Their Applications, Academic Press, New York, NY, USA,

Tan BK, Wu RS (1993). Nonlinear Rossby waves and their interactions. I. Collision of envelope solitary Rossby waves. Sci. China B 36:1367–1380.

Tang XY, shukla PK (2007). Lie symmetry analysis of the quantum Zakharov equations, Phys. Scrip. 76:665–668. http://dx.doi.org/10.1088/0031-8949/76/6/013

Weiss J, Tabor M, Carnevale G (1983). The Painlev'e property for partial differential equations. J. Math. Phys. 24:522-526.

Wang ML, Li X, Zhang J (2008). The $\left(\frac{G'}{G}\right)$ expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys. Lett. A 372:417-423. http://dx.doi.org/10.1016/j.physleta.2007.07.051

Yusufoglu E (2008). New solitary for the MBBM equations using Exp-function method. Phys. Lett. A 372:442-446. http://dx.doi.org/10.1016/j.physleta.2007.07.062

Yusufoglu E, Bekir A (2008). Exact solutions of coupled nonlinear Klein-Gordon equations. Math. Computer Modelling, 48:1694-1700. http://dx.doi.org/10.1016/j.mcm.2008.02.007

Zayed EME Gepreel KA (2009). The $\left(\frac{G'}{G}\right)$ expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics. J. Math. Phys. P. 50, 013502-013512. http://dx.doi.org/10.1063/1.3033750

Zayed EME (2009). The $\left(\frac{G'}{G}\right)$ expansion method and its applications to some nonlinear evolution equations in the mathematical physics, J. Appl. Math. Comput. 30:89-103. http://dx.doi.org/10.1007/s12190-008-0159-8

Zayed EME (2010). Traveling wave solutions for higher dimensional nonlinear evolution equations using the $\left(\frac{G'}{G}\right)$ expansion method. J. Appl. Math. Inform. 28:383-395.

Zhang S, Sun YN, Dong JMB (2011). The modified $\left(\frac{G'}{G}\right)$ expansion method for nonlinear evolution equations. Z. Naturforsch. 66a:33-39. http://dx.doi.org/10.5560/ZNA.2011.66a0033

Zayed EME, Abdelaziz MAM (2012). The two variables $\left(\frac{G'}{G},\frac{1}{G}\right)$ expansion method for solving the nonlinear KdV-mKdV equation. Math. Prob. Eng. Article ID 725061. 14 Pages.

Zayed EME, Hoda Ibrahim SA, Abdelaziz MAM (2012). Traveling wave solutions of the nonlinear (3+1)-dimensional Kadomtsev-Petviashvili equation using the two variables $\left(\frac{G'}{G},\frac{1}{G}\right)$ expansion method. J. Appl. Math. Article ID 560531. 8 Pages.

Zayed EME, Alurrfi KAE (2014a). The $\left(\frac{G'}{G},\frac{1}{G}\right)$ expansion method and its applications to find the exact solutions of nonlinear PDEs for nanobiosciences. Math. Prob. Eng, Article ID 521712. 10 Pages.

Zayed EME, Alurrfi KAE (2014b). The $\left(\frac{G'}{G},\frac{1}{G}\right)$ expansion method and its applications for solving two higher order nonlinear evolution equations. Math. Prob. Engineering, Article ID 746538. 20 Pages.

Yajima N, Satsuma J (1979). Soliton solutions in a diatomic lattice system. Prog. Theor. Phys. 62:370–378. http://dx.doi.org/10.1143/PTP.62.370

Zhang S, Xia TC (2008). A further improved tanh-function method exactly solving the (2+1) -dimensional dispersive long wave equations. Appl. Math. E-Notes, 8:58-66.

Zhang S, Tong JL,Wang W (2008). A generalized $\left(\frac{G'}{G}\right)$ expansion method for the mKdV equation with variable coefficients. Phys. Lett. A, 372:2254-2257. http://dx.doi.org/10.1016/j.physleta.2007.11.026

Zhang S (2008). Application of Exp-function method to high-dimensional nonlinear evolution equations. Chaos, Solitons and Fractals, 38:270-276. http://dx.doi.org/10.1016/j.chaos.2006.11.014

academicJournals

Vol. 9(19), pp. 430-437, 16 October, 2014
DOI: 10.5897/IJPS2013.4015
ISSN 1992 - 1950
Article Number:B1E1DEC48333

http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

A novel method to develop an automobile assembly line system

Ali A. J. Adham* and Ahmad N. N. Kamar

Faculty of Industrial Management, University Malaysia Pahang, Gambang, Pahang, 26300, Malaysia.

Received 25 August 2013; Accepted 7 October, 2014

The assembly line is an important component of the automobile production process. The function of the assembly line is to produce different models of vehicles with minimum work in the process. For better performance, activities on the assembly line should be performed to minimise the process steps and achieve other objectives. This study develops a new dynamic sequencing method to improve activities on the assembly line and also an automated sequence-control system. Three methods, namely the Multi-Objectives Model, the Genetic Algorithm System and the Simulation Model, are integrated to enhance the efficiency of the assembly line by controlling the processing time within the workstations. The results show that the method was able to improve the working time performance and also increase throughputs.

Key words: Processing time, assembly line, mathematical method, genetic algorithm system, simulation model.

INTRODUCTION

Body Shop (BS), Paint Shop (PS), Assembly Shop (AS) and Test Shop (TS) and other sub-assemblies, also called stations, have the function of feeding the main assembly. The production of automobiles in the AS is a typical example of the mixed-model production system (Wonjoon and Hyunoh, 1997). Figure 1 shows the assembly system of the automobile production system. In the figure, all the assembly plants have their own stations (namely S_1, S_2….S_n). The sub-assembly stations are also shown in the figure.

In the automobile industry system, one of the areas under consideration is the Assembly Line Balancing Problem (ALBP) which distributes the total workload among manufacturing stages (Adham, 2012; Ali and Razman, 2011; Toshio et al., 1996). There were many researchers who studied the issues related to the ALBP and the Production Line System (PLS) in order to obtain the best solution (Razam and Ali, 2012; Minh and Soemon, 2008; Williams, 2007).

The Hybrid Model (HM), combining the Multi-Objectives Model (MOM), the Genetic Algorithm System (GAS) and the Simulation Model (SM), is presented in this study. It is a new technique and one of the most powerful methods to obtain the best balance of the cycle. Many real-world

*Corresponding author. E-mail: aliadham12@yahoo.com

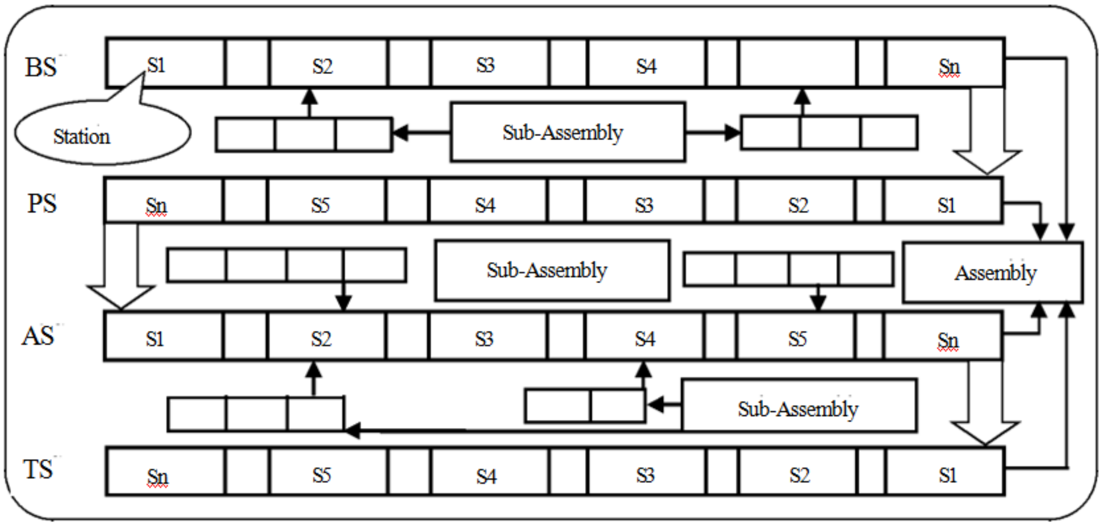

Figure 1. Assembly line system (ALS).

problems require an optimal solution that could be obtained by adopting the HM approach. The HM developed in this study is able to assist managers to have an optimal cycle time (CT), balancing the ALS and managing the production plan to know the capacity of the assembly line after solving the ALBP (Ali and Razman, 2012; Amir and Farhad, 2006; Anand et al., 2012).

The contribution of this study is to approach an integrated model (including the MOM and the GAS) to solve the queuing problem within the stations on the assembly line, with the SM to manage the capacity of the assembly line. Additionally, the integrated model can combine the unbalanced assembly line problem and the ratio of the production plan. As a result, the method will achieve the target by minimising unbalanced CT and maximising workload to achieve the production plan. This study focuses on the main problem of the production line which is balancing CT within the stations. The unbalancing problem occurs when not all stations are able to complete all tasks at the same time (Christian and Armin, 2009). As a result, it causes a congestion problem on the production line and the resources are underutilised. Figure 2 presents the ALBP which is unbalanced CT within the stations on the assembly line.

METHODOLOGY

Multi-objectives model (MOM)

The MOM is formulated to create a balanced time within stations through to obtaining optimal balance within the stations. There are two goals of the MOM: (1) to minimise the queuing time within the stations; (2) to minimise the idle time within the stations.

$$\text{Min } Q = \max \sum_{i=1}^{p} \sum_{i=1}^{sq} |QU_{ij}| X_{ij} \qquad (1) \quad (1^{st} \text{ goal})$$

$$\text{Min } Id = \max \sum_{i=1}^{p} \sum_{j=1}^{sd} DT_{ij} X_{ij} \qquad (2) \quad (2^{nd} \text{ goal})$$

Where: **Q:** total queuing time within the stations, **QU:** queuing between the stations, **Id:** total idle time within the stations, **DT:** idle time between the stations.

The MOM aids management to achieve either the optimum solution (Razman and Ali, 2011; Razman and Ali, 2010). Figure 3 shows the implementation of the MOM for the ALS. The MOM will reduce the queuing and the idle time to obtain the best balance, then the best solution for the PLS.

GAS

The GAS is formulated to create an advanced balance time within stations through shuffling of the tasks in order to obtain an optimum balance. The GAS will select the task that should be moved within stations according to the objectives (1) and (2). Figure 4 presents the model of moving tasks among the stations. In the figure, there are two categories of task movement: the first category is a movement (1) from station (1) towards the station (n) passing through all stations respectively. The second category is movement (2) from station (n) towards station (1) passing through all stations respectively. The aim is to find the best solution for these objects using the GAS. As seen, the solution should allow all points to be passed by choosing the closest path among them in one go. The task movement occurs after selection of the first task with a high CT from any station, which should be moved towards the next station which has a low CT and function as a final task. Otherwise, the GAS will select the final task with high CT from any station and move it towards the previous station with low CT and have it function as a first task. This formulation of GAS will be more realistic, that is, create an optimum balance of CT within stations.

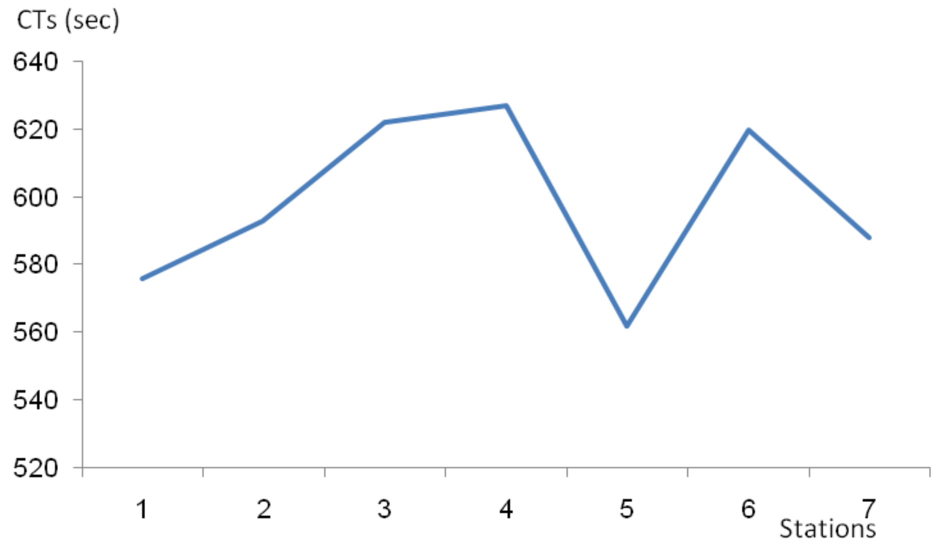

Figure 2. Unbalanced CTs.

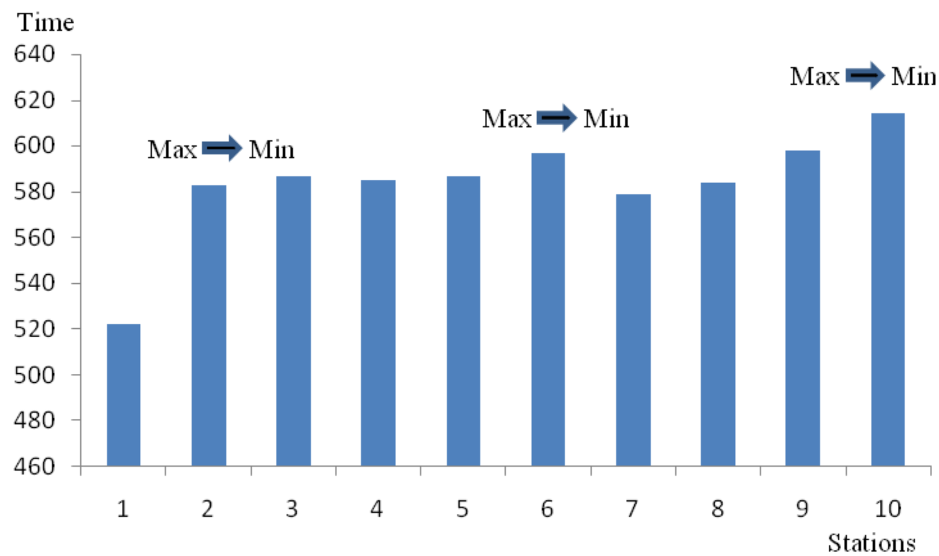

Figure 3. Unbalancing on the assembly line.

Genetic algorithm objectives

The GAS approach in this study aims to achieve two goals: rebalancing CT within the stations for each shop, through task movement; and also redistributing the jobs among the workers to obtain the optimum solution. The GAS obtains the optimum balance (optimum solution) of the ALS with two objectives which are:

(i) First goal: Moving the tasks among the stations.
(ii) Second goal: After applying the MOM, if the ALS still has queuing issues, the GAS will redistribute the jobs to the workers in order to achieve the optimum balance.

The formulas of the GAS are presented in Equations (3) and (4). These explain how the GAS achieves a time balance on the assembly line.

$$Goal1 = CTS_1 \approx CTS_2 \approx CTS_3 \approx CTS_4 \approx,,,, \approx CTS_n$$ (3) (1st goal)

$$Goal2 = RJS_1 \approx RJS_2 \approx RJS_3 \approx RJS_4 \approx,,,, \approx RJS_n$$ (4) (2nd goal)

Where: CTS_i =CT for each station, RJS_i = ratio of the jobs of each station.

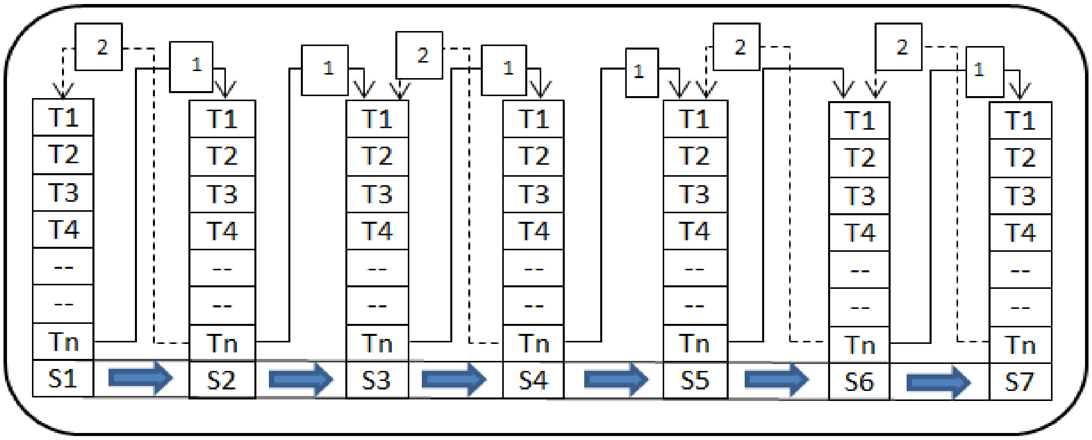

Figure 4. Task movement among the stations.

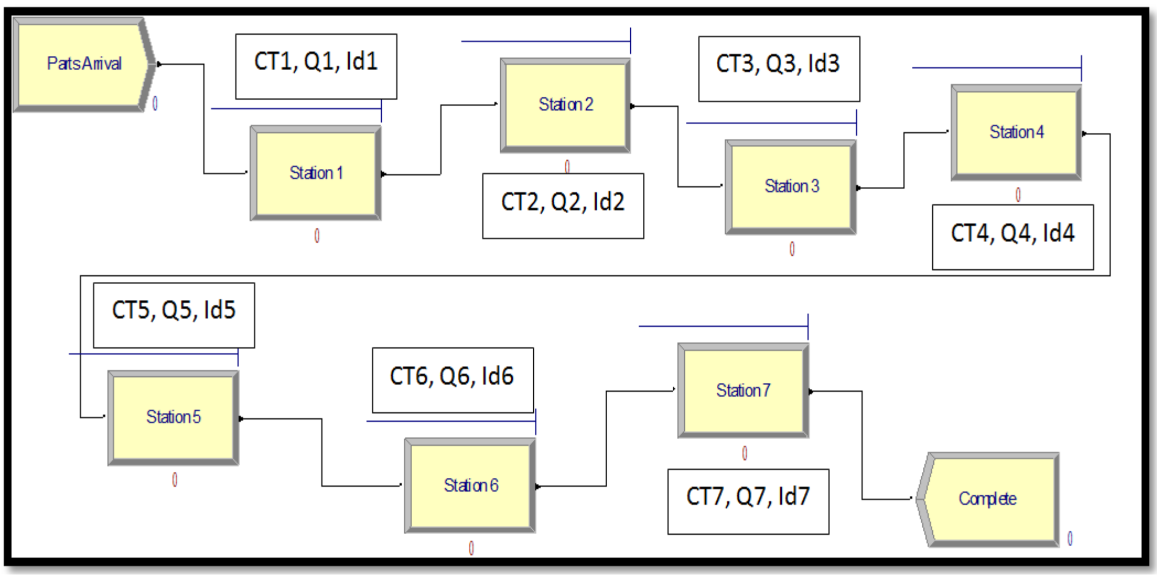

Figure 5. SM of ALS.

To achieve goal 1, the GAS should move the tasks among stations until it gets the best CT balance. This goal corresponds to the first and second objectives of MOM (1, 2), that is, to obtain the optimum balance.

Simulation model (SM)

Simulation is a technique with which a real-world problem can be mimicked and modelled with the aid of computers. The SM provides analysis and allows users to perform 'what-if' analysis where users can test different strategies or policies and observe how the model behaves before implementing it in the real world. Besides that, the simulation also serves as a training and educational tool (Holst and Bolmsjo, 2001).

In this study, the SM is developed using the ARENA simulation package. Figure 5 shows the SM of ALS. The chassis section in the ALS is modelled and inputs such as arrival time and processing time are incorporated into the model.

Hybrid model (HM)

HM flowchart

The HM, applied to solve both problems, which are queued and idle time, also manages a new plan depending on the available total working time to obtain the best balancing by applying the MOM. The SM will create new plans depending on the efficiency of the cycle time. Figure 6 shows the flowchart of the HM.

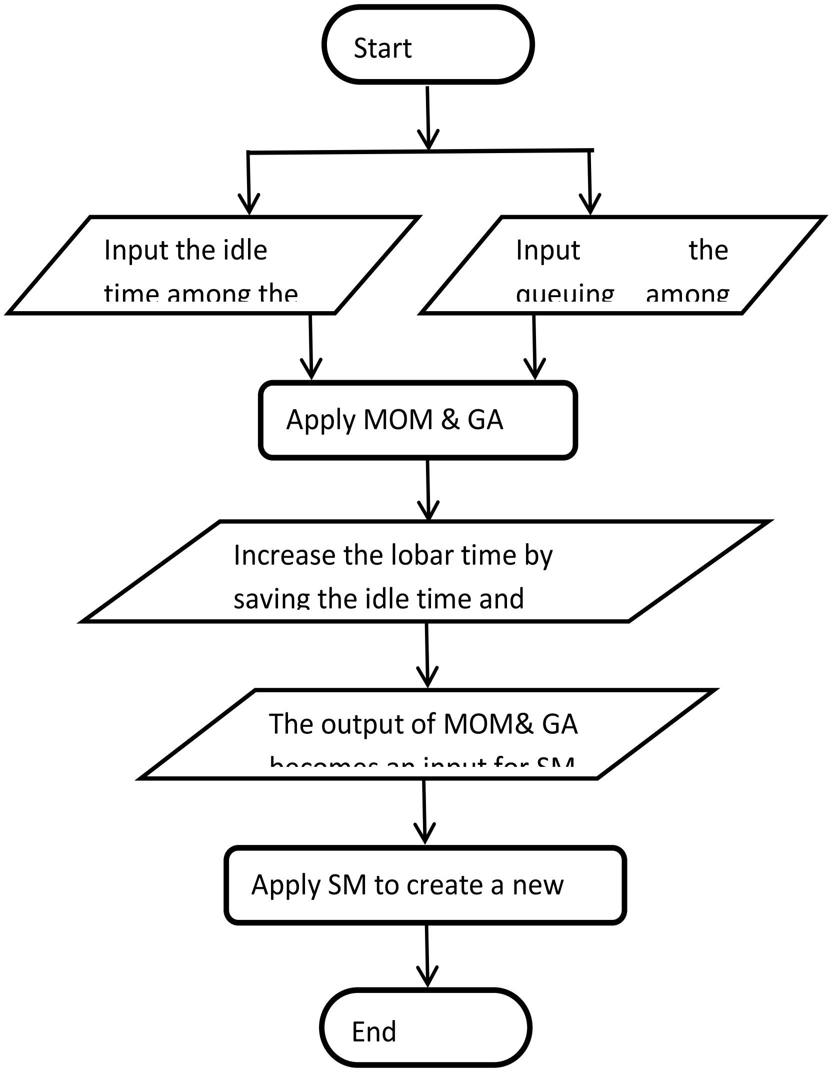

Figure 6. HM flowchart.

Procedure of HM

Computer software helps to optimise the ALS by applying the MOM and GAS. MATLAB software is used to solve the system issues to obtain the optimum balanced CT of the ALS. The procedure is as follows:

```
Procedure: solve the unbalanced CT of the assembly line.
Input data: CT, task number, number of workers for each station, queuing and idle time.
Output data: the optimum balancing of the ALS.
Begin
{
        Calculate the CT on the production line for each station
                While i < total number of stations
                        Balance process time tasks
                Move the tasks among the stations
        i=i+1
End
Print the optimum balance
} End
```

Once the optimum balancing of ALS is calculated using the MOM, a SM is developed. The SM is constructed to test the maximum number of cars that can be produced if the ALS CT is balanced in order to optimise the capacity of the production line.

MODEL RESULTS

Balancing problems of the ALS

CT is the time taken to complete all tasks at the stations of the shops. For a highly efficient ALS, the CT should be equal among the stations (Nai-Chieh and I-Ming, 2011).

Table 1. Variables of the chassis section.

Stations	CT seconds (L3)	No. tasks (L2)	Workers (L3)
1	576	122	2
2	593	116	2
3	622	130	1
4	627	125	2
5	562	118	1
6	620	127	1
7	588	120	1
Total	4,188	858	10

Table 2. Queuing and idle time at the CS.

No. station	CT before applying the model	CT after applying the model	Queuing before	Idle time before	Queuing after	Idle time after
CS_1	576	593	17	0	6	0
CS_2	593	599	29	0	0	0
CS_3	622	599	5	0	4	0
CS_4	627	603	0	65	0	1
CS_5	562	602	58	0	0	6
CS_6	620	596	0	32	0	0
CS_7	588	596	0	0	0	0
	4,188	4,188	109	97	10	7

Normally, it is a very challenging target to reach a balance of CT within the stations. An unbalanced CT is caused by the queuing problem on the assembly line. This study examined the chassis section which is one part of the assembly line.

Chassis section (CS)

The CS has seven stations. Its function is to assemble an engine, bellows, axle and other mechanical works. The total processing time in this section is 4,188 s. Table 1 describes the operational aspects of this data. It shows that station 1 has a processing time of 576 s with 122 tasks. Only two workers are involved at this station. Table 2 shows the queuing time and the idle time before and after applying the MOM. The queuing and the idle time before applying the model were 109 and 97 s, respectively. After applying the HM, the queuing and idle time become 10 and 7 s, respectively. As a result, the model reduced time arising from the queuing problem by around 99 s (1.65 min), and 90 s (1.5 min) due to idle time. The total time saved is 189 s (3.15 min) in preparation of only one car.

Figure 7 presents the CT station before and after applying the MOM and the GAS to the ALS. In the figure, L1 represents the working CT before applying the model; L2 represents the best balancing within stations after applying the model.

Simulation results

The current assembly line operates for 7.5 h per day and produces 28 cars daily. The SM is used to test the maximum number of cars that can be produced if the assembly line operates according to the optimum CT calculated using the MOM. Table 3 shows the number of cars that can be produced daily. The simulation results reveal that by adopting the new balanced CT, the ALS can produce an additional four cars daily, given that the maximum number of operating hours of the ALS is 7.5 h. Producing more than 32 cars will require additional working hours. This finding serves as a guideline on how many additional cars can be produced daily without exceeding the current capacity and maximum duration.

DISCUSSION

The new method combines the MOM and the GAS with the SM to solve the unbalancing and planning

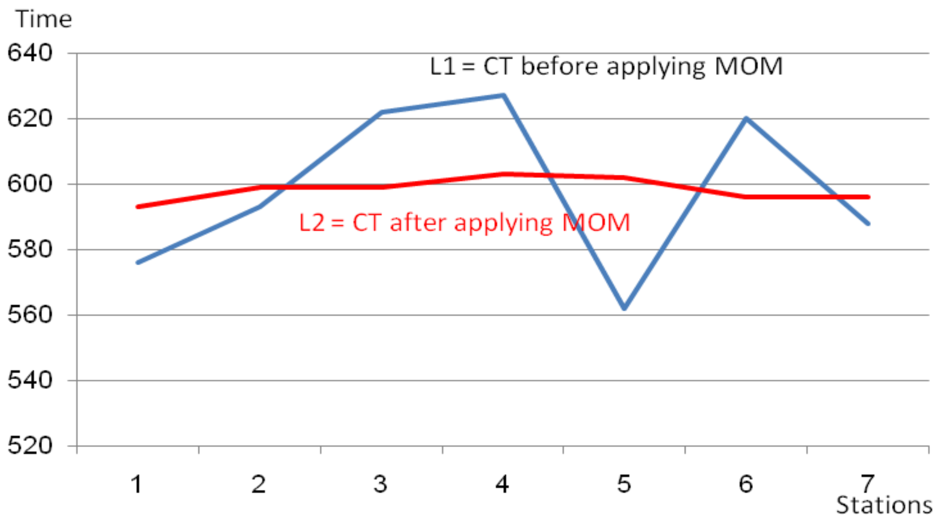

Figure 7. CTs before applying, after applying the MOM and the GAS.

Table 3. Number of cars produced and hours needed.

Number of cars produced	Hours
20	5.69
24	6.02
28	6.36
30	6.69
32	7.03
34	7.36
36	7.53
37	7.70

problems in the ALS. This method was applied to the chassis section of the ALS to obtain the optimum balance and plan. The MOM the GAS saved 189 s in the chassis section. The total queuing before applying the MOM and the GAS was 109 s (1.81 min); it became 10 s (0.16 min) after applying the MOM and GAS which saved 99 s (1.65 min) for preparation of one car. Also, the model reduced the idle time within the stations as well. It was 97 s (1.61 min) before applying the model and became 7 s (0.11 min) for preparation of one car. Therefore, the total time saved by using the MOM and the GAS is 189 s (3.15 min) for preparation of one car in respect to both issues of the queuing and the idle time. Besides minimising queuing and idle time, this study further enhanced the results by developing a SM to test the maximum number of cars that can be produced daily by the ALS if the balanced CTs are adopted. The results show that from the current number of 28 cars produced, the balanced ALS can produce a maximum of 32 cars daily. The new technique is beneficial to all workshops and sections of the production line as it increases the capacity of the production line in automobile manufacture.

Conclusion

The ALS is very important for the automobile industry. The unbalancing variations within stations are difficult problems which affect efficiency of the assembly line. This study proposed a new technique to solve the problems, such as queuing and idle time within stations. The HM combines the MOM, the GAS and the SM to obtain the optimum solution and plan. As a result, the new technique is very important for enhancement of the efficiency of the assembly line. Moreover, the HM reduces the unbalanced time within the stations and increases production by four cars per day.

Conflict of Interest

The authors have not declared any conflict of interest.

ACKNOWLEDGMENTS

This paper was supported by the University of Malaysia Pahang, under Grant No. RDU120380. The authors would like to thank the university and all staff that assisted in the success of this research. Also, thanks to anonymous referees whose insights improved the content of this manuscript.

REFERENCES

Adham AAJ (2012). An integrated model for production line balancing planning, PhD dissertation, University Malaysia Pahang, Pahang, Malaysia.

Ali AJA, Razman M (2012). Enhancing efficiency of automobile assembly line using the fuzzy logical and multi-objective genetic algorithm. WCCI 2012 IEEE World Congress on Computational Intelligence June, 10-15, Brisbane, Australia.

Ali AJA, Razman M (2011). Process queuing in the automobile manufacturing body shop: Using a multi-objective model. Int. J. Phys. Sci. 6(30):6928-6933.

Amir A, Farhad KA (2006). Multi-objective lead time control problem in multistage assembly systems using an interactive method. J. Appl. Math. Computation. 176:609-620.

Williams A (2007). Product service systems in the automobile industry: contribution to system innovation. J. Cleaner Production. 15:1093-1103.

Christian B, Armin S (2009). Balancing assembly lines with variable parallel workplaces: problem definition and effective solution procedure. Eur. J. Operational Res. 199:359-374.

Holst L, Bolmsjo G (2001). Simulation integration in manufacturing system development: A study of Japanese industry. Ind. Manage. Data Syst. 101(7):339-35.

Anand MSA, Sarkar S, Rajendra S (2012). Application of distributed control system in automation of process industries. Int. J. Emerging Technol. Adv. Eng. Website 6(2):2250-2459.

Minh DN, Soemon T (2008). Emergence of simulations for manufacturing line designs in Japanese automobile manufacturing plants. Proceeding of the Winter Simulation Conference IEEE. Japan.

Nai-Chieh W, I-Ming C (2011). A solution procedure for type e simple assembly line balancing problem. J. Comput. Industrial Eng. 61:824–830.

Razman M, Ali AAJ (2011). Optimum efficiency of production line in automobile manufacture. Afr. J. Bus. Manage. 6(20):6266-6275.

Razman MT, Ali AJ (2010). Design and analysis of automobiles manufacturing system based on simulation. Canadian Center Sci. Edu. J. Model Appl. Sci. 4(7):130-134.

Toshio S, Atsushi N, Kouske S, Takashi M, Nobuyoshi H (1996). Development of the new human-conscious automobile assembly plant. Annals of the CIRP. 46(1):381-384.

Wonjoon C, Hyunoh S (1997). A real-time sequence control system for the level production of the automobile assembly line. Comput. Ind. Eng. 33(3-4):769-772.

International Journal of Physical Sciences

Related Journals Published by Academic Journals

- *African Journal of Pure and Applied Chemistry*
- *Journal of Internet and Information Systems*
- *Journal of Geology and Mining Research*
- *Journal of Oceanography and Marine Science*
- *Journal of Environmental Chemistry and Ecotoxicology*
- *Journal of Petroleum Technology and Alternative Fuels*